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# Indeterminate Equation

## Assessment

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*Math for Gifted Students*

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Assessment

# Indeterminate Equation



## Instructions

- *Write down and submit intermediate steps along with your final answer.*
- *If the final result is too complex to compute, give the expression. e.g.  $C_{100}^{50}$  is acceptable.*
- *Problems are not necessarily ordered based on their difficulty levels.*
- *Always ask yourself what makes this problem a good one to practise?*
- *Complete the My Record section below before submission.*

## My Comments and Notes

Assessment

# Indeterminate Equation



## Practice 1

Find all ordered integer pairs  $(x, y)$  such that  $x + xy + y = 8$ .

## Practice 2

Solve in integers the equation  $41x + 37y = 13$ .

## Practice 3

Solve in positive integers the following equations:

(i)  $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$

(ii)  $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$

(iii)  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}$

## Practice 4

Solve the equation  $x^2 + y^2 = 6x - 4y - 13$ .

## Practice 5

How many ordered integer pairs  $(x, y)$  are there such that  $5(x^2 + 3) = y^2$  ?

## Practice 6

Find all the right triangles that satisfy the following two conditions:

(i) the lengths of all its three sides are integers, and

(ii) its area and perimeter are numerically equal

# Indeterminate Equation



## Practice 7

Solve in positive integers the equation  $y^2 = x^2 + x + 1$ .

## Practice 8

Find all pairs of positive integers  $(x, y)$  where  $x$  and  $y$  are relatively prime, such that the following expression is an integer:

$$\frac{x}{y} + \frac{15y}{4x}$$

## Practice 9

Solve in integers the equation:  $x^2 + y^2 = 2015$ .

## Practice 10

Find all positive integer triplets  $(x, y, z)$  such that  $3^x + 4^y = 5^z$ .

## Practice 11

Solve in integers the equation  $x^3 + 2y^3 = 4z^3$ .

## Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.

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# Reference Solutions

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## Indeterminate Equation



## Practice 1

Find all ordered integer pairs  $(x, y)$  such that  $x + xy + y = 8$ .



*Tip: The factorization method and integer divisibility.*

*Solution 1: Polynomial Factorization*

The given equation can be re-written as:

$$(x + 1)(y + 1) = 9$$

Because both  $x$  and  $y$  are integers, both  $(x + 1)$  and  $(y + 1)$  are integers too. It follows that they must be paired divisors of 9:

$x + 1$	$y + 1$	$x$	$y$
1	9	0	8
3	3	2	2
9	1	8	0
-1	-9	-2	-10
-3	-3	-4	-4
-9	-1	-10	-2

Therefore we conclude there are totally 6 solutions.

*Solution 2: Integer Divisibility*

Rearrange the given equation as an equation with respect to  $x$ :  $(y + 1)x = 8 - y$ . Hence:

$$x = \frac{8 - y}{y + 1} = \frac{9 - (1 + y)}{y + 1} = \frac{9}{y + 1} - 1 \quad (1)$$

Because  $x$  is an integer,  $\frac{9}{y + 1}$  must be an integer. This follows that  $(y + 1)$  must be a divisor of 9, or

$$\begin{aligned} y + 1 &= -9, -3, -1, 1, 3, 9 \\ y &= -10, -4, -2, 0, 2, 8 \end{aligned}$$

Setting these values to *Equation 1*, respectively, leads to

$$x = -2, -4, -10, 8, 2, 0$$

## Indeterminate Equation



### Practice 2

Solve in integers the equation  $41x + 37y = 13$ .



*Tip: The Euclidean method, and the  $ax + by = 1$  and  $ax + by = c$  patterns.*

First, let's solve

$$41x + 37y = 1 \quad (2)$$

This can be done by the Euclidean method. We have

$$41 = 37 \times 1 + 4$$

$$37 = 4 \times 9 + 1$$

Therefore

$$1 = 37 - 4 \times 9 = 37 - (41 - 37 \times 1) \times 9 = -41 \times 9 + 37 \times 10$$

This means *Equation 2* has one solution  $(-9, 10)$ , and its general solution is given by:

$$\begin{cases} x = -9 + 37t \\ y = 10 - 41t \end{cases} \quad (3)$$

where  $t$  is an integer parameter.

It follows that

$$\begin{cases} x = -9 \times 13 + 37 \times 13t = -117 + 481t \\ y = 10 \times 13 - 41 \times 13t = 130 - 533t \end{cases} \quad (4)$$

is the solution to the original question  $41x + 37y = 13$ .

### Practice 3

Solve in positive integers the following equations:

(i)  $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$

(ii)  $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$

(iii)  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5}$

## Indeterminate Equation



(i)



*Tip: The factorization method, and the  $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$  pattern.*

The given equation can be rewritten as  $(x - 3)(y - 3) = 9$ . Therefore both  $(x - 3)$  and  $(y - 3)$  must be divisors of 16. Without loss of generality, let's assume  $x - 3 \geq y - 3$ . This follows that one of the following must hold:

$$\begin{cases} x - 3 = 9 \\ y - 3 = 1 \end{cases} \quad \text{or} \quad \begin{cases} x - 3 = 3 \\ y - 3 = 3 \end{cases}$$

Solving these two systems lead to  $(x, y) = (12, 4), (6, 6)$ . Hence all the solutions are

$$(x, y) = (12, 4), (6, 6), (4, 12)$$

(ii)



*Tip: The inequality method, and the  $\frac{1}{x} + \frac{1}{y} = \frac{m}{n}$  pattern.*

By symmetry, let's assume  $x \leq y$ . Hence

$$\frac{1}{x} \geq \frac{1}{y}$$

It follows that,

$$\frac{1}{x} \geq \frac{1}{2} \times \frac{5}{6} = \frac{5}{12} \quad (5)$$

or  $x \leq 2$ .

Testing  $x = 1, 2$  respectively finds  $(2, 3)$  is one solution. Therefore all the solutions to the given equations are

$$(x, y) = (2, 3), (3, 2)$$



*Quiz: Can you use this method to solve (i) above?*

(iii)



*Tip: The inequality method, and the  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{m}{n}$  pattern.*

By the symmetrical argument, let's assume  $0 < x \leq y \leq z$ . It follows:

$$\frac{1}{x} < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3}{x}$$

Assessment

# Indeterminate Equation



Then  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{5} \implies \frac{1}{x} < \frac{3}{5} \leq \frac{3}{x} \implies 2 \leq x \leq 5.$

Now we proceed with casework:

If  $x = 2$ , then  $\frac{1}{y} + \frac{1}{z} = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}.$

If  $x = 3$ , then  $\frac{1}{y} + \frac{1}{z} = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}.$

If  $x = 4$ , then  $\frac{1}{y} + \frac{1}{z} = \frac{3}{5} - \frac{1}{4} = \frac{7}{20}.$

If  $x = 5$ , then  $\frac{1}{y} + \frac{1}{z} = \frac{3}{5} - \frac{1}{5} = \frac{2}{5}.$

The equivalent equation in every case is in the form of:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n} \quad \text{or} \quad \frac{1}{x} + \frac{1}{y} = \frac{m}{n}$$

They can all be solved by using the techniques that are presented in (i) and (ii) above. Solving these equations leads to the following solutions under the assumption  $0 < x \leq y \leq z$ .

(2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 10), and (5, 5, 5).

Therefore, all the solutions are just distinct permutations of the above set.

## Practice 4

Solve the equation  $x^2 + y^2 = 6x - 4y - 13$ .



*Tip: The sum of square method.*

The given equation is equivalent to:

$$(x - 3)^2 + (y + 2)^2 = 0$$

Because squares cannot be negative, the only possibility to make this equation hold is  $(x, y) = (3, -2)$ .

# Indeterminate Equation



## Practice 5

How many ordered integer pairs  $(x, y)$  are there such that  $5(x^2 + 3) = y^2$  ?



*Tip: Property of square numbers.*

It is obvious that  $(x^2 + 3)$  must be a multiple of 5. It follows that the unit digit of  $(x^2 + 3)$  must be either 0 or 5. Equivalently,  $x^2$  must end with 8 or 3. However no square number can end with 8 or 3. Hence this equation is not solvable.

## Practice 6

Find all the right triangles that satisfy the following two conditions:

- (i) the lengths of all its three sides are integers, and
- (ii) its area and perimeter are numerically equal



*Tip: The Pythagorean triplet formula.*

By the Pythagorean triplet formula, the lengths of three sides can be written as:

$$\begin{cases} x = m^2 - n^2 \\ y = 2mn \\ z = m^2 + n^2 \end{cases}$$

where  $m$  and  $n$  are two positive integers.

If its area and perimeter equal in values, the following must hold:

$$\frac{1}{2}(m^2 - n^2)(2mn) = (m^2 - n^2) + 2mn + (m^2 + n^2)$$

It follows that:

$$\begin{aligned} (m^2 - n^2)(mn) &= 2m^2 + 2mn \\ (m + n)(m - n)mn &= 2m(m + n) \\ (m - n)n &= 2 \end{aligned}$$

Assessment

# Indeterminate Equation



This is a basic indeterminate equation that can be solved using the factorization method.

$$\begin{cases} m - n = 1 \\ n = 2 \end{cases} \quad \text{or} \quad \begin{cases} m - n = 2 \\ n = 1 \end{cases}$$

We find  $(m, n) = (3, 2)$  or  $(3, 1)$ .

Setting these values into the Pythagorean triplet formula produces two such triangles: 5-12-13 and 6-8-10.

## Practice 7

Solve in positive integers the equation  $y^2 = x^2 + x + 1$ .



*Tip: The squeeze method.*

From the given equation and the condition  $x > 0$ , it is easy to see that

$$y^2 < x^2 < (x + 1)^2$$

Note that  $x$  and  $(x + 1)$  are two consecutive integers. Therefore it is impossible to have another integer whose square is between  $x^2$  and  $(x + 1)^2$ .

Hence, we conclude that no solution is possible.

## Practice 8

Find all pairs of positive integers  $(x, y)$  where  $x$  and  $y$  are relatively prime, such that the following expression is an integer:

$$\frac{x}{y} + \frac{15y}{4x}$$



*Tip: The quadratic method.*

Let  $u = \frac{x}{y}$ , then the given problem is equivalent to:

$$u + \frac{15}{4u} = k$$

where  $k$  is an integer. It is obvious that  $u$  is a positive rational number because both  $x$  and  $y$  are positive integers.

Assessment

## Indeterminate Equation



Rewriting this relationship leads to:

$$4u^2 - 4ku + 15 = 0 \quad (6)$$

Because *Equation 6* is solvable in rational number, its discriminant must be a square number. Let

$$\Delta = 16k^2 - 4 \times 4 \times 15 = n^2$$

where  $n$  is an integer. Or:

$$16k^2 - n^2 = 240$$

Clearly,  $n^2$  is a multiple of  $16 = 4^2$ , setting  $n = 4m$  leads to:

$$\begin{aligned} 16k^2 - 16m^2 &= 240 \\ k^2 - m^2 &= 15 \\ (k + m)(k - m) &= 15 \end{aligned} \quad (7)$$

*Equation 7* can be solved by the factorization method. Because both  $k$  and  $m$  are positive integers, we have  $k + m > k - m$ . Consequently, one of the two systems must hold:

$$\begin{cases} k + m = 15 \\ k - m = 1 \end{cases} \quad \text{or} \quad \begin{cases} k + m = 5 \\ k - m = 3 \end{cases}$$

Solving the above two systems leads to

$$(k, m) = (8, 7), (4, 1)$$

Setting  $k = 8$  to the quadratic formula of *Equation 6*:

$$u = \frac{4 \times 8 \pm \sqrt{(4 \times 8)^2 - 4 \times 4 \times 15}}{2 \times 4} = 4 \pm \frac{7}{2} = \frac{15}{2}, \frac{1}{2}$$

Setting  $k = 4$  leads:

$$u = \frac{4 \times 4 \pm \sqrt{(4 \times 4)^2 - 4 \times 4 \times 15}}{2 \times 4} = 2 \pm \frac{1}{2} = \frac{5}{2}, \frac{3}{2}$$

Therefore, we conclude there are four solutions:

$$(x, y) = (15, 2), (1, 2), (5, 2) \text{ and } (3, 2)$$

## Indeterminate Equation



## Practice 9

Solve in integers the equation:  $x^2 + y^2 = 2015$ .



*Tip: Number theory / Frequently used MOD conclusions.*

Taking MOD 4 on both sides leads to

$$x^2 + y^2 = 2015 \equiv 3 \pmod{4}$$

However this relationship cannot hold. Therefore the original equation is not solvable in integers.

## Practice 10

Find all positive integer triplets  $(x, y, z)$  such that  $3^x + 4^y = 5^z$ .



*Tip: The MOD method*

First, let's show  $x$ ,  $y$ , and  $z$  must be all even.

Taking (mod 4) on both sides of the equation leads to:

$$(-1)^x + 0 \equiv 1^z \pmod{4}$$

Clearly, this relationship can only hold if  $x$  is even.

Next, taking (mod 3) on both sides yields:

$$0 + 1^y \equiv (-1)^z \pmod{3}$$

Therefore,  $z$  must be even too.

As such, let  $x = 2k$ ,  $z = 2p$ , and note  $4^y = (2^y)^2$ , the original equation becomes:

$$(3^k)^2 + (2^y)^2 = (5^p)^2$$

Therefore  $(3^k, 2^y, 5^p)$  forms a Pythagorean triplet. Hence, there exist positive integers  $m$  and  $n$  such that <sup>1</sup>:

$$\begin{cases} 3^k = m^2 - n^2 \\ 2^y = 2mn \\ 5^p = m^2 + n^2 \end{cases}$$

<sup>1</sup>Note  $3^k$  is an odd number. Therefore it cannot equal  $2mn$

# Indeterminate Equation



Because  $2^y = 2mn$ , both  $m$  and  $n$  must be some power of 2. Let  $m = 2^t$  and  $n = 2^s$  where  $t$  and  $s$  are non-negative integers satisfying  $t + s = y - 1$ . Note  $m > n \implies t > s$ .

It follows that:

$$\begin{cases} 3^k = m^2 - n^2 = 2^{2t} - 2^{2s} = 2^{2s}(2^{2(t-s)} - 1) \\ 5^p = m^2 + n^2 = 2^{2t} + 2^{2s} = 2^{2s}(2^{2(t-s)} + 1) \end{cases}$$

Because neither  $3^k$  nor  $5^p$  is divisible by 2, we conclude  $2^{2s}$  must equal 1. This means  $s = 0$ , and  $2^{2(t-s)} = 4$  or  $t = 1$ . It is followed by  $k = p = 1$ .

Hence, the given equation has only one positive integer solution:  $x = y = z = 2$ .

## Practice 11

Solve in integers the equation  $x^3 + 2y^3 = 4z^3$ .



*Tip: The infinite descent method.*

If there exists such a positive integer solution  $(x, y, z)$ , then  $x$  must be even. Let  $x = 2x_1$ :

$$\begin{aligned} (2x_1)^3 + 2y^3 &= 4z^3 \\ 4x_1^3 + y^3 &= 2z^3 \end{aligned}$$

This means  $y$  must be even too. Let  $y = 2y_1$ :

$$\begin{aligned} 4x_1^3 + (2y_1)^3 &= 2z^3 \\ 2x_1^3 + 4y_1^3 &= z^3 \end{aligned}$$

This in turn shows  $z$  is also even. Let be  $z = 2z_1$ :

$$\begin{aligned} 2x_1^3 + 4y_1^3 &= (2z_1)^3 \\ x_1^3 + 2y_1^3 &= 4z_1^3 \end{aligned}$$

This last equation is in the same form of the original one. Hence, we conclude if  $(x, y, z)$  is a positive integer solution,  $x$ ,  $y$ , and  $z$  must be all even, and  $(x_1, y_1, z_1) = (\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$  will be a solution too. It is clear that the process of  $(x, y, z) \implies (x_1, y_1, z_1)$  is repeatable. Therefore an infinitive decreasing solution series can be constructed, which is impossible by the principle of infinite descent. Thus, the equation is unsolvable in positive integers.

## Indeterminate Equation



## Practice 12

Find all the triangles whose sides are three consecutive integers and areas are also integers.



*Tip: The Pell's equation.*

Let three sides' lengths be  $z - 1$ ,  $z$ , and  $z + 1$ , respectively. Then by the Heron's formula, the triangle's area is given by:

$$\begin{aligned} S &= \sqrt{\frac{3}{2}z \times \left(\frac{3}{2}z - (z - 1)\right) \left(\frac{3}{2}z - z\right) \left(\frac{3}{2}z - (z + 1)\right)} \\ &= \frac{z}{4} \sqrt{3(z^2 - 4)} \end{aligned} \quad (8)$$

If  $S$  is an integer, then  $3(z^2 - 4)$  must be a square number. Let

$$3(z^2 - 4) = 3w^2$$

In addition, from *Equation 8*, it is clear that  $z$  must be even because, otherwise, both  $z$  and  $\sqrt{3(z^2 - 4)}$  will be odd. This will make  $S$  a non-integer.

Letting  $z = 2x$  leads to  $4x^2 - 4 = 3w^2$ . This means that  $w$  must be even too. Letting  $w = 2y$  and simplifying yield:

$$x^2 - 3y^2 = 1$$

This is a Pell's equation. Its fundamental solution is

$$(x, y) = (2, 1)$$

and general solution is given by:

$$\begin{cases} x_n = \frac{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n}{2} \\ y_n = \frac{(2 + \sqrt{3})^n - (2 - \sqrt{3})^n}{2\sqrt{3}} \end{cases} \quad (9)$$

As a result, there exist infinitely many such triangles. Three sides are  $(2x_n - 1, 2x_n, 2x_n + 1)$  where  $x_n$  is given by *Equation 9*, and area is  $3 \cdot x_n \cdot y_n$ .

The three smallest such triangles can be obtained by setting  $n = 1, 2$ , and  $3$ , respectively:

$$(3, 4, 5), (13, 14, 15), (51, 52, 53)$$

## Indeterminate Equation



## Answer Keys

Practice 1:  $(x, y) = (0, 8), (8, 0), (2, 2), (-2, -10), (-10, -2), (-4, -4)$

Practice 2:

$$\begin{cases} x = -9 \times 13 + 37 \times 13t = -117 + 481t \\ y = 10 \times 13 - 41 \times 13t = 130 - 533t \end{cases}$$

where  $t$  is an integer parameter.

Practice 3:

(i)  $(x, y) = (2, 3), (3, 2)$

(ii)  $(x, y) = (12, 4), (6, 6), (4, 12)$

(iii) Permutation of the following sets:  $(2, 11, 110), (2, 12, 60), (2, 14, 35), (2, 15, 30), (2, 20, 20), (3, 4, 60), (3, 5, 15), (3, 6, 10), (4, 4, 10),$  and  $(5, 5, 5)$ .

Practice 4:  $(x, y) = (3, -2)$

Practice 5: No solution exists.

Practice 6: 5-12-13 and 6-8-10.

Practice 7: No solution exists.

Practice 8:  $(x, y) = (15, 2), (1, 2), (5, 2)$  and  $(3, 2)$

Practice 9: No solution exists.

Practice 10:  $(x, y, z) = (2, 2, 2)$

Practice 11: No solution exists.

Practice 12: There exist infinitely many such triangles. Refer to the reference solution.