Square Numbers

Math All Star Practice by Subject Series

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How to get the most out of practice

The most important tip is to learn before practice. This approach is what students do in schools. It also applies to competition math. Systematic learning will help students develop the following skills, and get the most out of practice afterwards.

1. Being able to recognize which subject a given problem belongs to.
2. Knowing relevant solving techniques for such type of problems.
3. Being able to choose the most appropriate solution for this particular problem.

There are many tutorial materials available on the website `https://www.mathallstar.org`, including books, videos, articles, and so on.

Meanwhile, it is important to read and understand reference solution instead of just checking the answer. One objective of practice is for students to check whether they understand and master all the necessary solving techniques or not. However, merely obtaining the correct answer does not necessarily mean the most suitable technique is used. Therefore, it is beneficial to understand the solution in addition to obtaining the correct answer.

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Chapter 1

Review

1.1 Ending Digit Property

Let's begin with one of the simplest properties of square numbers.

Theorem 1.1.1 Ending Digit

A square number can only end with 0, 1, 4, 5, 6, or 9. In another word, a square number cannot end with 2, 3, 7, or 8.

This property is self evident.

1.2 Divisor Property

Theorem 1.2.1 Divisor

A positive integer is a square if and only if it has odd number of divisors.

This is also obvious because all divisors come in pair. A positive integer becomes a square if one paired divisors are the same.

1.3 Remainder Properties

Among all the remainder properties, MOD 4 is the most used.

Theorem 1.3.1 MOD 4

Let n^2 be a square number, then $n^2 \equiv 0, 1 \pmod 4$.

Proof

If n is odd, let $n = 2k + 1$. Then $n^2 = (2k + 1)^2 = 4(k^2 + k) + 1 \equiv 1 \pmod 4$.

If n is even, let $n = 2k$. Then $n^2 = (2k)^2 = 4k^2 \equiv 0 \pmod 4$.

QED

This theorem can immediately lead to the following useful conclusion.

Example 1.3.1

Let x and y be two integers, show that

$$x^2 + y^2 \not\equiv 3 \pmod 4$$

Another application of the MOD 4 property is to judge the last 2 digits of a square number. This is because the remainder of a number when being divided by 4 is determined by its last two digits. Here is one practice that utilizes this property:

Example 1.3.2

If a square number's tens digit is 7, what is its units digit?

MOD 8 and MOD 16 can also be used to investigate a square number's last 3 and 4 digits because the divisibilities of 8 and 16 are determined by the last 3 and 4 digits.

Example 1.3.3

Show that if n is a square number, then $n \equiv 0, 1, 4 \pmod 8$.

Proof

If $n \equiv 0 \pmod 8$, then $n^2 \equiv 0 \pmod 8$.

If $n \equiv \pm 1 \pmod 8$, then $n^2 \equiv 1 \pmod 8$.

If $n \equiv \pm 2 \pmod 8$, then $n^2 \equiv 4 \pmod 8$.

If $n \equiv \pm 3 \pmod 8$, then $n^2 \equiv 1 \pmod 8$.

If $n \equiv 4 \pmod 8$, then $n^2 \equiv 0 \pmod 8$.

Summarizing these results leads to the claim.

QED

Example 1.3.4

Show that if n is a square number, then $n \equiv 0, 1, 4, 9 \pmod{16}$.

Proof

If $n \equiv 0 \pmod{16}$, then $n^2 \equiv 0 \pmod{16}$.

If $n \equiv \pm 1 \pmod{16}$, then $n^2 \equiv 1 \pmod{16}$.

If $n \equiv \pm 2 \pmod{16}$, then $n^2 \equiv 4 \pmod{16}$.

If $n \equiv \pm 3 \pmod{16}$, then $n^2 \equiv 9 \pmod{16}$.

If $n \equiv \pm 4 \pmod{16}$, then $n^2 \equiv 16 \equiv 0 \pmod{16}$.

If $n \equiv \pm 5 \pmod{16}$, then $n^2 \equiv 25 \equiv 9 \pmod{16}$.

If $n \equiv \pm 7 \pmod{16}$, then $n^2 \equiv 49 \equiv 1 \pmod{16}$.

If $n \equiv 8 \pmod{16}$, then $n^2 \equiv 64 \equiv 0 \pmod{16}$.

QED

1.4 Other Fun Facts

These fun facts will be left as practice problems. They can all be proved using the properties described in previous sections.

- Let N be an odd square number. Then N's tens digit must be even.
- Let N be a square number. If its units digit is 6, then its tens digit must be odd.
- Let N be a square number. If its tens digit is odd, then its units digit must be 6.
- Let N be a square number. If its units digit is neither 4 nor 6, then its tens digit must be even.

Other fun facts:

- The differences of consecutive terms in a series of square numbers is consecutive odd numbers. i.e.

$$0^2, 1^2, 2^2, 3^2, 4^2 \cdots \implies 1, 3, 5, 7, \cdots$$

Chapter 2

Practice

Practice 1

Let n^2 be a square number. Show that $n^2 \equiv 0, \pm 1 \pmod 5$. Or, in another word, the remainder of a square number when being divided by 5 can only be 0, 1, or -1 (equivalent to 4).

(ref: 4156)

Practice 2

Find the smallest positive integer n such that $\frac{12!}{n}$ is a square.

(ref: 2392)

Practice 3

Let k be a positive integer, show that $4k+3$ cannot be a square number.

(ref: 2360)

Practice 4

Find the number of integer pairs (x, y) such that $x^2 + y^2 = 2015$.

(ref: 4140)

Practice 5

How many terms in this sequence are squares?

$$1, 11, 111, 1111, \cdots$$

(ref: 4147)

Practice 6

How many terms in this sequence are squares?

$$4, 44, 444, 4444, \cdots$$

(ref: 4148)

Practice 7

How many numbers in this infinitive series are squares? $1, 14, 144, 1444, 14444, \cdots$

(ref: 2361)

Practice 8

For any given positive integer n, prove $(n^2 + n + 1)$ cannot be a perfect square.

(ref: 1430)

Practice 9

Solve the following equation in positive integers: $15x - 35y + 3 = z^2$

(ref: 2363)

Practice 10

Solve the following equation in positive integers: $3 \times (5x + 1) = y^2$

(ref: 2365)

Practice 11

Solve in positive integers $5 \times (x^2 + 3) = y^2$.

(ref: 2366)

Practice 12

Find such a positive integer n such that both $(n - 100)$ and $(n - 63)$ are square numbers.

(ref: 2390)

Practice 13

Find such a positive integer n such that both $(n + 23)$ and $(n - 30)$ are square numbers.

(ref: 2391)

Practice 14

Find a 4-digit square number x such that if every digit of x is increased by 1, the new number is still a perfect square.

(ref: 2626 - China)

Practice 15

If a square number's tens digit is 7, what is its units digit?

(ref: 4141)

Practice 16

Let N be an odd square number. Show that N's tens digit must be even.

(ref: 4149)

Practice 17

Let N be a square number. If its units digit is 6, then its tens digit must be odd.

(ref: 4150)

Practice 18

Let N be a square number. If its tens digit is odd, then its units digit must be 6.

(ref: 4151)

Practice 19

Let N be a square number. If its units digit is neither 4 nor 6, then its tens digit must be even.

(ref: 4152)

Practice 20

Find all positive integer n such that n is a square and its last four digits are the same.

(ref: 2362)

Practice 21

Find a four-digit square number whose first two digits are the same and the last two digits are the same too.

(ref: 2364)

Practice 22

Let A and B be two positive integers and $A = B^2$. If A satisfies the following conditions, find the value of B:

- A's thousands digit is 4
- A's tens digit is 9
- The sum of all A's digits is 19

(ref: 2382)

Practice 23

If the middle term of three consecutive integers is a perfect square, then the product of these three numbers is called a *beautiful* number. What is the greatest common divisor of all the *beautiful* numbers?

(ref: 2384 - China)

Practice 24

Find the smallest square whose last three digits are the same but not equal 0.

(ref: 2385)

Practice 25

Mary found a 3-digit number that, when multiplied by itself, produced a number which ended in her original 3-digit number. What is the sum of all the numbers which have this property?

(ref: 2820)

Practice 26

Find a square number which has two thousand and eighteen 6s and some numbers of 0s?

(ref: 2901)

Practice 27

There are 100 light bulbs lined up in a row in a long room. Each bulb has its own switch and is currently switched off. The room has an entry door and an exit door. There are 100 people lined up outside the entry door. Each bulb is numbered consecutively from 1 to 100. So is each person. Person No. 1 enters the room, switches on every bulb, and exits. Person No. 2 enters and flips the switch on every second bulb (turning off bulbs 2, 4, 6...). Person No. 3 enters and flips the switch on every third bulb (changing the state on bulbs 3, 6, 9...). This continues until all 100 people have passed through the room.

How many of the light bulbs are illuminated after the 100^{th} person has passed through the room?

(ref: 4142)

Practice 28

A positive integer n is said to be good if there exists a perfect square whose sum of digits in base 10 is equal to n. For instance, 13 is good because $7^2 = 49$ and $4 + 9 = 13$. How many good numbers are among $1, 2, 3, \cdots, 2007$?

(ref: 2816)

Chapter 3

Solution

Practice 1

Let n^2 be a square number. Show that $n^2 \equiv 0, \pm 1 \pmod 5$. Or, in another word, the remainder of a square number when being divided by 5 can only be 0, 1, or -1 (equivalent to 4).

(ref: 4156)

If $n \equiv 0 \pmod 5$, then $n^2 \equiv 0 \pmod 5$.

If $n \equiv \pm 1 \pmod 5$, then $n^2 \equiv 1 \pmod 5$.

If $n \equiv \pm 2 \pmod 5$, then $n^2 \equiv 4 \equiv -1 \pmod 5$.

Therefore, we conclude the claim holds.

Practice 2

Find the smallest positive integer n such that $\frac{12!}{n}$ is a square.

(ref: 2392)

We factorize 12! as $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$. Therefore, the answer is

$$3 \cdot 7 \cdot 11 = \boxed{231}$$

Practice 3

Let k be a positive integer, show that $4k + 3$ cannot be a square number.

(ref: 2360)

Because $4k + 3$ is odd, it can only be a square of an odd number. But an odd number's square must have a remainder of 1 when being divided by 4. Therefore, the claim holds.

Practice 4

Find the number of integer pairs (x, y) such that $x^2 + y^2 = 2015$.

(ref: 4140)

The answer is $\boxed{0}$.

This is because $2015 \equiv 3 \pmod 4$. But there exists no integer solution to $x^2 + y^2 \equiv 3 \pmod 4$.

Practice 5

How many terms in this sequence are squares?

$$1, 11, 111, 1111, \cdots$$

(ref: 4147)

The answer is $\boxed{1}$.

All these terms are odd. In order for any one to be a square, it must have a remainder of 1 when being divided by 4. Only the first term 1 satisfies this condition. And clearly, 1 is a square. Hence, the answer is 1.

Practice 6

How many terms in this sequence are squares?

$$4, 44, 444, 4444, \cdots$$

(ref: 4148)

The answer is $\boxed{1}$.

All terms are even. Therefore, in order for any one to be a square, its remainder when being divided by 16 must be either 0 or 4. Only the first term satisfy this requirement. And indeed 4 is a square. (Note that the remainder when being divided by 16 is determined by its last three digits. So we only need to check the first three terms.)

This problem can also be reasoned using the conclusion of practice 5. Because 4 is a square number, therefore dividing every term of this sequence by 4 will not change the fact whether that term is a square. By practice 5, we know there is only one term in

$$1, 11, 111, 1111, \cdots$$

is square. So there is only one square in this sequence as well.

Practice 7

How many numbers in this infinitive series are squares? $1, 14, 144, 1444, 14444, \cdots$

(ref: 2361)

Among the first three terms, 1 and 144 are obviously square, but 14 is not.

We claim that none of the rest terms will be a perfect square. This can be verified by using the MOD 16 rule. All of them will have a remainder of 12 when divided by 16. (Note that the remainder when being divided by 16 is determined by the last three digits.)

Therefore, the answer to this question is $\boxed{2}$.

Practice 8

For any given positive integer n, prove $(n^2 + n + 1)$ cannot be a perfect square.

(ref: 1430)

Because $n^2 < n^2 + n + 1 < (n + 1)^2$ and it is impossible to have another square between squares of two consecutive integers, we can safely conclude that $(n^2 + n + 1)$ cannot be a square.

Practice 9

Solve the following equation in positive integers: $15x - 35y + 3 = z^2$

(ref: 2363)

Clearly, $(15x - 35y)$ is a multiple of 5 which means that $(15x - 35y + 3)$ will end with either 3 or 8. Therefore it cannot be a square number. This means the given equation has no integer solution.

Practice 10

Solve the following equation in positive integers: $3 \times (5x + 1) = y^2$

(ref: 2365)

No solution exists.

If x is odd, the left side will end with 8 which cannot be square. If x is even, the left side will end with 3 which cannot be square either.

Practice 11

Solve in positive integers $5 \times (x^2 + 3) = y^2$.

(ref: 2366)

This equation has no solution. This is because, as a square number, x^2 can only end with digit 0, 1, 4, 5, 6, and 9. This means that the unit digit of $(x^2 + 3)$ can only be 3, 4, 7, 8, 9, and 2. it follows that $(x^2 + 3)$ is not a multiple of 5. Now, we find the left side contains only one divisor of 5. Hence, it cannot be a square number as the right side requires.

Practice 12

Find such a positive integer n such that both $(n - 100)$ and $(n - 63)$ are square numbers.

(ref: 2390)

Let $n - 100 = m^2$ and $n - 63 = k^2$, then we have

$$(n - 63) - (n - 100) = k^2 - m^2 \implies 37 = (k + m)(k - m)$$

Because 37 is a prime number which can only be factorized as 1×37, Hence we must have

$$k - m = 1, k + m = 37 \implies k = 19, m = 18$$

It follows that $n = m^2 + 100 = \boxed{424}$.

Practice 13

Find such a positive integer n such that both $(n + 23)$ and $(n - 30)$ are square numbers.

(ref: 2391)

Let $n + 23 = m^2$ and $n - 30 = k^2$, then we have

$$(n + 23) - (n - 30) = m^2 - k^2 \implies 53 = (m + k)(m - k)$$

Because 53 is a prime number which can only be factorized as 1×53, Hence we must have

$$m - k = 1, m + k = 53 \implies k = 26, m = 27$$

It follows that $n = m^2 - 23 = \boxed{706}$.

Practice 14

Find a 4-digit square number x such that if every digit of x is increased by 1, the new number is still a perfect square.

(ref: 2626 - China)

Let this 4-digit number be N, then $N = k^2$ for some integer k. Increasing every digital of N by 1 is the same as adding 1111 to N. Hence, $(N + 1111)$ is also a square. Let it be m^2. Subtracting these two relations leads to

$$(N + 1111) - N = m^2 - k^2 \implies 1111 = (m + k)(m - k)$$

There are only two ways to express 1111 as a multiple of two integers: 1×1111 and 11×101. Therefore we have

$$m + k = 1111, m - k = 1$$

or

$$m + k = 101, m - k = 11$$

Solving these two systems and noting k^2 is a 4 digit-number leads to only one solution $k = 45$ which means $N = \boxed{2025}$.

Practice 15

If a square number's tens digit is 7, what is its units digit?

(ref: 4141)

The answer is $\boxed{6}$.

A square number can only end with 0, 1, 4, 5, 6, and 9. Hence, the candidate of last two digits are 70, 71, 74, 75, 76, and 79.

Meanwhile, a square number must satisfy $n^2 \equiv 0, 1 \pmod 4$. Among these candidates, only 76 meets this criteria. Therefore, we conclude that the answer is 6.

Practice 16

Let N be an odd square number. Show that N's tens digit must be even.

(ref: 4149)

Because N is an odd square, it must end with 1, 5, or 9. Meanwhile, the following relation must also hold:

$$N \equiv 1 \pmod 4$$

However, it is easy to see that none of 11, 15, 19, 31, $\cdots$ satisfies this property, but all of 21, 25, $\cdots$ do. Hence, we can conclude this claim hold.

(Note: 1, 5, 9 differs by 4, and 10, 30, 50, $\cdots$ differs by 20. Both of them are multiples of 4. Therefore, we just need to check the first term 11. If it does not satisfy $N \equiv 1 \pmod 4$, none of them will do.)

Practice 17

Let N be a square number. If its units digit is 6, then its tens digit must be odd.

(ref: 4150)

Clearly, N is even. Hence, N must be a multiple of 4. Then, it is easy to verify that its tens digit must be odd: 16, 36, 56, 76, and 96 are all multiples of 4. But none of 06, 26, 46, 66, and 86 satisfies this requirement.

Practice 18

Let N be a square number. If its tens digit is odd, then its units digit must be 6.

(ref: 4151)

First, by practice 16, its unit digit cannot be odd. Therefore, N must be even. It follows that its unit digit can only be 0, 4, or 6.

Next, because N is even, then $N \equiv 0 \pmod 4$ must hold. It is now easy to verify only 6 is possible when the tens digit is odd because none of 10, 30, $\cdots$, and 14, 34, $\cdots$ is a multiple of 4.

Practice 19

Let N be a square number. If its units digit is neither 4 nor 6, then its tens digit must be even.

(ref: 4152)

First, by practice 16, if its unit digit is odd, its tens digit must be even.

When N is an even square but ends with neither 4 nor 6, then it must end with 0. Meanwhile, an even square must be a multiple of 4. Therefore, its tens digit must be even in order to satisfy this requirement.

Practice 20

Find all positive integer n such that n is a square and its last four digits are the same.

(ref: 2362)

Clearly, any square number ending with 0000 qualifies. We are going to show no other solution exists.

First, we filter by square number's MOD 4 property: $n^2 \equiv 0, 1 \pmod 4$.

If n ends with 1111, 3333, 5555, 7777, or 9999, it must be a square of an odd number which means n must be a multiple of 4 plus 1. Therefore, among these candidates only 3333 and 7777 may be possible. However, neither of them is possible because no square number can end with 3 or 7.

If n ends with 2222, 4444, 6666, or 8888, it must be a square of an even number which means n must be a multiple of 4. Therefore, among these candidates, only 4444 and 8888 are possible. However, 8888 is impossible because no square number can end with 8.

Next, we can apply the MOD 16 property, $n^2 \equiv 0, 1, 4, 9 \pmod{16}$ to eliminate 4444.

Therefore, we conclude the last four digits can only be 0000.

Practice 21

Find a four-digit square number whose first two digits are the same and the last two digits are the same too.

(ref: 2364)

Let this number be $n = \overline{aabb} = \overline{a0b} \times 11$.

If n is a square, then it must have even number of divisor 11. Hence, it must hold that $11 \mid \overline{a0b}$. By the division by 11 property, we must have $11 \mid a + b$. Therefore, there are 8 possibilities:

$$(a, b) = (2, 9), (9, 2), (3, 8), (8, 3), (4, 7), (7, 4), (5, 6), (6, 5)$$

By square number's ending digit property, only $(a, b) = (2, 9)$, $(7, 4)$, $(5, 6)$, and $(6, 5)$ are possible. By practice 16, the last two digits of a square cannot be both odd. Therefore, neither $(2, 9)$ nor $(6, 5)$ is possible. Additionally, by practice 17, $(5, 6)$ cannot be a square either. This leaves $(7, 4)$ the only possible choice.

Because $\boxed{7744} = 88^2$, there it is the only solution to this problem.

Practice 22

Let A and B be two positive integers and $A = B^2$. If A satisfies the following conditions, find the value of B:

- A's thousands digit is 4
- A's tens digit is 9
- The sum of all A's digits is 19

(ref: 2382)

First, let's consider the last two digits of A. Because its tens digit is odd, its units digit must be 6 (see practice 18). Now, the sum of its digits is already 19. Hence, its hundreds digit must be 0. In fact, we find $4096 = 64^2$ indeed is a square. Hence, $B = \boxed{64}$.

Practice 23

If the middle term of three consecutive integers is a perfect square, then the product of these three numbers is called a *beautiful* number. What is the greatest common divisor of all the *beautiful* numbers?

(ref: 2384 - China)

Firstly, the smallest beautiful number equals $3 \times 4 \times 5 = 60$. Therefore the answer cannot be greater than 60. We are going to show that the answer is $\boxed{60}$. This is equivalent to showing that $60 \mid (n^2-1)n^2(n^2+1)$ holds for any positive integer $n \geq 2$.

Write $(n^2-1)n^2(n^2+1)$ as $\Big((n-1)n(n+1)\Big)\Big(n(n^2+1)\Big)$. Its first part, $(n-1)n(n+1)$ is a product of three positive integers, therefore, it must be a multiple of 3!. For the second part $n(n^2+1)$, either n or (n^2+1) will be even. Therefore it is a multiple of 2. Together, the whole thing must be a multiple of $3! \times 2 = 12$.

Next, we are going to show it is also a multiple of 5. By the conclusion of practice 1, $n^2 \equiv 0, \pm 1 \pmod 5$. Therefore, one of (n^2-1), n^2, and (n^2+1) must be a multiple of 5.

Therefore, $(n^2-1)n^2(n^2+1)$ must be a multiple of $12 \times 5 = 60$. This means the greatest common divisor of all beautiful numbers is 60.

Practice 24

Find the smallest square whose last three digits are the same but not equal 0.

(ref: 2385)

Firstly, this must be an even number. This is because if its tens digit is odd, then the units digit must be 6 (see practice 18). Therefore, the last two digits cannot be the same.

Because an even square must be a multiple of 4 and cannot end with 8, the last two digits can only be 44. This means the last three digits can only be 444.

However, 444 is not a square but $1444 = 38^2$ is. Therefore the answer is $\boxed{1444}$.

Practice 25

Mary found a 3-digit number that, when multiplied by itself, produced a number which ended in her original 3-digit number. What is the sum of all the numbers which have this property?

(ref: 2820)

If square of a 3-digit number ends with itself, then the square of its last two digits must end with this two digit number as well. There are only two such numbers with this property: 25 and 76. Therefore this three number must end with 25 or 76

Only one 3-digit number which ends in 25 whose square ends with itself: 625.

One 3-digit number which ends in 76 whose square ends with itself: 376.

Therefore, the sum of these two numbers is $625 + 376 = \boxed{1001}$.

Practice 26

Find a square number which has two thousand and eighteen 6s and some numbers of 0s?

(ref: 2901)

It is impossible.

If the unit digit is 6, then the tens digit must be an odd number.

If the unit digit is not 6, then this number must end with even numbers of 0. Removing these 0s, we end up in the previous situation.

Practice 27

There are 100 light bulbs lined up in a row in a long room. Each bulb has its own switch and is currently switched off. The room has an entry door and an exit door. There are 100 people lined up outside the entry door. Each bulb is numbered consecutively from 1 to 100. So is each person. Person No. 1 enters the room, switches on every bulb, and exits. Person No. 2 enters and flips the switch on every second bulb (turning off bulbs 2, 4, 6...). Person No. 3 enters and flips the switch on every third bulb (changing the state on bulbs 3, 6, 9...). This continues until all 100 people have passed through the room.

How many of the light bulbs are illuminated after the 100^{th} person has passed through the room?

(ref: 4142)

The answer is 10. All the lights whose labels are square numbers are on. The rest are off.

The light will be on if and only if its switch has been turned for an odd number of times. This means its label should have odd number of divisors. Only square numbers meet this condition.

Practice 28

A positive integer n is said to be good if there exists a perfect square whose sum of digits in base 10 is equal to n. For instance, 13 is good because $7^2 = 49$ and $4 + 9 = 13$. How many good numbers are among $1, 2, 3, \cdots, 2007$?

(ref: 2816)

If a positive integer is a multiple of 3, then its square is a multiple of 9, and so is the sum of the digits of its square.

If a positive integer is not a multiple of 3, then its square is 1 more than a multiple of 3, and so is the sum of the digits of its square.

Therefore, we only need to study two categories of candidates.

The square of $\underbrace{9\cdots9}_{m}$ is $\underbrace{9\cdots9}_{m-1}8\underbrace{0\cdots0}_{m-1}1$. Its digit sum is $9m$. Therefore, all multiples of 9 are good. There are $2007 \div 9 = 223$ of them not exceeding 2007.

The square of $\underbrace{3\cdots3}_{m}5$ is $\underbrace{1\cdots1}_{m}\underbrace{2\cdots2}_{m+1}5$ Its digit sum is $3m+7$. Since 1 and 4 are also good, all numbers 1 more than a multiple of 3 are good, and there are $2007 \div 3 = 669$ of them.

Hence there are altogether $223 + 669 = \boxed{992}$ good numbers not exceeding 2007.