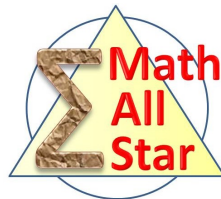

Assessment

Power Calculation



Math for Gifted Students

<http://www.mathallstar.org>

Assessment

Power Calculation


Practice 1

Compute the sums of the following expressions:

i) $1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + 2016 \times 2017 \times 2018$

ii) $\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{2016 \times 2017 \times 2018}$

Practice 2

Let

$$f(r) = \sum_{j=2}^{2016} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2016^r}$$

Find the value of

$$\sum_{k=2}^{\infty} f(k)$$

Practice 3

Find the values of the following nested radicals:

i) $\sqrt{5 + \sqrt{5^2 + \sqrt{5^4 + \sqrt{5^8 + \dots}}}}$

ii) $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \dots}}}}$

Practice 4

Without using a calculator, explain why the following approximation holds:

$$\sqrt{20 + \sqrt{20 + \sqrt{20}}} - \sqrt{20 - \sqrt{20 - \sqrt{20}}} \approx 1$$

Power Calculation



Practice 5

Find the length of the leading non-repeating block in the decimal expansion of $\frac{2017}{3 \times 5^{2016}}$. For example the length of the leading non-repeating block of $\frac{1}{6} = 0.1\bar{6}$ is 1.

Practice 6

Simplify:

i) $C_n^0 + 2C_n^1 + 4C_n^2 + \dots + 2^n C_n^n$

ii) $C_n^1 + 2C_n^2 + 3C_n^3 + \dots + nC_n^n$

iii) $C_n^0 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \dots + \frac{1}{n+1}C_n^n$

iv) $(C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 \dots + (C_n^n)^2$

Practice 7

Simplify:

i) $\sin \theta + 2 \sin 2\theta + 3 \sin 3\theta \dots + n \sin n\theta$

ii) $\sin \theta + \frac{1}{2} \cdot \sin 2\theta + \frac{1}{4} \cdot \sin 3\theta + \dots$

Practice 8

Without using a calculator, find the value of $\cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{9\pi}{13}$.

Reference Solutions

Power Calculation



Practice 1

Compute the sums of the following expressions:

i) $1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + 2016 \times 2017 \times 2018$

ii) $\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{2016 \times 2017 \times 2018}$

Note: as a practice, it is sufficient to give the answer as those in boxes.

$$\begin{aligned}
 \text{i)} \quad & 1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + 2016 \times 2017 \times 2018 \\
 & = 3! \times \left(\frac{1 \times 2 \times 3}{3!} + \frac{2 \times 3 \times 4}{3!} + \cdots + \frac{2016 \times 2017 \times 2018}{3!} \right) \\
 & = 3! \times \left(C_3^3 + C_4^3 + \cdots + C_{2018}^3 \right) \\
 & = \boxed{3! \times C_{2019}^4} \\
 & = 4141845698256
 \end{aligned}$$

$$\begin{aligned}
 \text{ii)} \quad & \frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{2016 \times 2017 \times 2018} \\
 & = \frac{1}{2} \times \left(\frac{1}{1 \times 2} - \frac{1}{2 \times 3} \right) + \frac{1}{2} \times \left(\frac{1}{2 \times 3} - \frac{1}{3 \times 4} \right) + \cdots + \frac{1}{2} \times \left(\frac{1}{2016 \times 2017} - \frac{1}{2017 \times 2018} \right) \\
 & = \boxed{\frac{1}{2} \times \left(\frac{1}{1 \times 2} - \frac{1}{2017 \times 2018} \right)} \\
 & = \frac{508788}{2035153}
 \end{aligned}$$

Practice 2

Let

$$f(r) = \sum_{j=2}^{2016} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2016^r}$$

Find the value of

$$\sum_{k=2}^{\infty} f(k)$$

Power Calculation



$$\begin{aligned}
 \sum_{k=2}^{\infty} f(k) &= f(2) + f(3) + f(4) + \dots \\
 &= \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{2016^2} \\
 &+ \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{2016^3} \\
 &+ \frac{1}{2^4} + \frac{1}{3^4} + \dots + \frac{1}{2016^4} \\
 &+ \dots \\
 &= \left(\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots \right) \\
 &+ \left(\frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \dots \right) \\
 &+ \dots \\
 &+ \left(\frac{1}{2016^2} + \frac{1}{2016^3} + \frac{1}{2016^4} + \dots \right) \\
 &= \frac{1}{2^2} \times \frac{1}{1-1/2} + \frac{1}{3^2} \times \frac{1}{1-1/3} + \dots + \frac{1}{2016^2} \times \frac{1}{1-1/2016} \\
 &= \frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \dots + \frac{1}{2016 \times 2015} \\
 &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{2015} - \frac{1}{2016} \right) \\
 &= \boxed{\frac{2015}{2016}}
 \end{aligned}$$

Practice 3

Find the values of the following nested radicals:

i) $\sqrt{5 + \sqrt{5^2 + \sqrt{5^4 + \sqrt{5^8 + \dots}}}}$

ii) $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \dots}}}}$

i) First let's compute $\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$.

Let $S = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}} > 0$, we find $S^2 = 1 + S \implies S = \frac{\sqrt{5}+1}{2}$. Then

$$\sqrt{5 + \sqrt{5^2 + \sqrt{5^4 + \sqrt{5^8 + \dots}}}} = \sqrt{5} \times \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}} = \sqrt{5} \times \frac{\sqrt{5} + 1}{2} = \frac{5 + \sqrt{5}}{2}$$

Power Calculation



ii) Setting $n = 1$ and $x = 2$ in the Srinivasa Ramanujan identity below leads to the answer $\boxed{3}$.

$$x + n = \sqrt{n^2 + x\sqrt{n^2 + (x+n)\sqrt{n^2 + (x+2n)\sqrt{\dots}}}}$$

Practice 4

Without using a calculator, explain why the following approximation holds:

$$\sqrt{20 + \sqrt{20 + \sqrt{20}}} - \sqrt{20 - \sqrt{20 - \sqrt{20}}} \approx 1$$

The approximation holds because:

$$\sqrt{20 + \sqrt{20 + \sqrt{20}}} \approx \sqrt{20 + \sqrt{20 + \sqrt{20 + \sqrt{20 + \dots}}}} = 5$$

$$\sqrt{20 - \sqrt{20 - \sqrt{20}}} \approx \sqrt{20 - \sqrt{20 - \sqrt{20 - \sqrt{20 - \dots}}}} = 4$$

Practice 5

Find the length of the leading non-repeating block in the decimal expansion of $\frac{2017}{3 \times 5^{2016}}$. For example the length of the leading non-repeating block of $\frac{1}{6} = 0.1\bar{6}$ is 1.

It is easy to see that the given expression can be decomposed as

$$\frac{2017}{3 \times 5^{2016}} = \frac{A}{3} + \frac{B}{5^{2016}}$$

where both A and B are two constants.

It is clear that $\frac{A}{3}$ is a repeating decimal from the tenths digit. Meanwhile, $\frac{B}{5^{2016}}$ is a decimal of 2016 digits to the right of the decimal point. Therefore, the length of the leading non-repeating block is 2016.

Power Calculation



Practice 6

Simplify:

i) $C_n^0 + 2C_n^1 + 4C_n^2 + \dots + 2^n C_n^n$

ii) $C_n^1 + 2C_n^2 + 3C_n^3 + \dots + nC_n^n$

iii) $C_n^0 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \dots + \frac{1}{n+1}C_n^n$

iv) $(C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 \dots + (C_n^n)^2$

i) Setting $x = 2$ in the following identity gives the answer of $\boxed{3^n}$.

$$(1+x)^n = C_n^0 + C_n^1 x + C_n^2 x^2 + \dots + C_n^n x^n$$

ii) Let $S = C_n^1 + 2C_n^2 + 3C_n^3 + \dots + nC_n^n$. Then

$$\begin{aligned} S &= 0 \times C_n^0 + 1 \times C_n^1 + \dots + (n-1) \times C_n^{n-1} + n \times C_n^n \\ S &= n \times C_n^n + (n-1) \times C_n^{n-1} + \dots + 1 \times C_n^1 + 0 \times C_n^0 \end{aligned}$$

Therefore

$$2 \cdot S = n \times (C_n^0 + C_n^1 + \dots + C_n^{n-1} + C_n^n) = n \times 2^n \implies S = \boxed{n \times 2^{n-1}}$$

iii) By identity $\frac{1}{k+1}C_n^k = \frac{1}{n+1}C_{n+1}^{k+1}$, we have

$$\begin{aligned} &C_n^0 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \dots + \frac{1}{n+1}C_n^n \\ &= \frac{1}{n+1}C_{n+1}^1 + \frac{1}{n+1}C_{n+1}^2 + \dots + \frac{1}{n+1}C_{n+1}^{n+1} \\ &= \frac{1}{n+1} \times (C_{n+1}^1 + C_{n+1}^2 + \dots + C_{n+1}^{n+1}) \\ &= \frac{1}{n+1} \times (2^{n+1} - 1) \end{aligned}$$

iv) By Vandermonde identity:

$$\begin{aligned} &(C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 \dots + (C_n^n)^2 \\ &= C_n^0 C_n^n + C_n^1 C_n^{n-1} + C_n^2 C_n^{n-2} + \dots + C_n^n C_n^0 \\ &= \boxed{C_{2n}^n} \end{aligned}$$

Power Calculation



Practice 7

Simplify:

i) $\sin \theta + 2 \sin 2\theta + 3 \sin 3\theta \cdots + n \sin n\theta$

ii) $\sin \theta + \frac{1}{2} \cdot \sin 2\theta + \frac{1}{4} \cdot \sin 3\theta + \cdots$

i) Let $S = \sin \theta + 2 \sin 2\theta + 3 \sin 3\theta \cdots + n \sin n\theta$. Then

$$\begin{aligned} 2 \cdot \cos \theta \cdot S &= \sin 2\theta + 2(\sin 3\theta + \sin \theta) + \cdots + n(\sin(n+1)\theta + \sin(n-1)\theta) \\ &= 2 \cdot S + n \sin(n+1)\theta - (n+1) \sin n\theta \end{aligned}$$

Therefore

$$S = \frac{(n+1) \sin n\theta - n \sin(n+1)\theta}{2(1 - \cos \theta)}$$

ii) Let $z = \cos \theta + i \sin \theta$, then

$$\begin{aligned} &\left(\cos \theta + \frac{1}{2} \cdot \cos 2\theta + \frac{1}{4} \cdot \cos 3\theta + \cdots \right) + i \left(\sin \theta + \frac{1}{2} \cdot 2 \sin 2\theta + \frac{1}{4} \cdot \sin 3\theta + \cdots \right) \\ &= z + \frac{1}{2} \cdot z^2 + \frac{1}{4} \cdot z^3 + \cdots \\ &= z / \left(1 - \frac{1}{2} \cdot z \right) \\ &= \frac{2z}{2 - z} \\ &= \frac{2(\cos \theta + i \sin \theta)}{(2 - \cos \theta) - i \sin \theta} \\ &= \frac{2(\cos \theta + i \sin \theta)((2 - \cos \theta) + i \sin \theta)}{((2 - \cos \theta) - i \sin \theta)((2 - \cos \theta) + i \sin \theta)} \\ &= \frac{(\cdots) + i(4 \sin \theta)}{5 - 4 \cos \theta} \end{aligned}$$

Therefore the answer is $\frac{4 \sin \theta}{5 - 4 \cos \theta}$.

Assessment

Power Calculation



Practice 8

Without using a calculator, find the value of $\cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{9\pi}{13}$.

This problem requires the following identity:

$$\cos \frac{\pi}{2n+1} + \cos \frac{3\pi}{2n+1} + \cdots + \cos \frac{(2n-1)\pi}{2n+1} = \frac{1}{2}$$

Let $x = \cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{9\pi}{13}$ and $y = \cos \frac{5\pi}{13} + \cos \frac{7\pi}{13} + \cos \frac{11\pi}{13}$. Then

$$x + y = \frac{1}{2} \quad \text{by the identity above}$$

and

$$\begin{aligned} xy &= \left(\cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{9\pi}{13} \right) \left(\cos \frac{5\pi}{13} + \cos \frac{7\pi}{13} + \cos \frac{11\pi}{13} \right) \\ &= -\frac{3}{2} \times \left(\cos \frac{\pi}{13} - \cos \frac{2\pi}{13} + \cos \frac{3\pi}{13} - \cos \frac{4\pi}{13} + \cos \frac{5\pi}{13} - \cos \frac{6\pi}{13} \right) \\ &= -\frac{3}{2} \times \left(\cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} + \cos \frac{7\pi}{13} + \cos \frac{9\pi}{13} + \cos \frac{11\pi}{13} \right) \\ &= -\frac{3}{2} \times \frac{1}{2} \\ &= -\frac{3}{4} \end{aligned}$$

Therefore x and y are two roots of the following equation:

$$t^2 - \frac{1}{2} \cdot t - \frac{3}{4} = 0$$

Clearly $x > 0$. Solving the above equation leads to $x = \boxed{\frac{1 + \sqrt{13}}{4}}$.