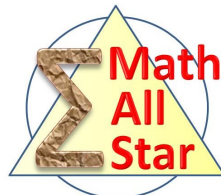


---

# Geometry

## Basic Trigonometry in Geometry

---



*Math for Gifted Students*

<http://www.mathallstar.org>




# Basic Trigonometry in Geometry



## Instructions

- Write down and submit intermediate steps along with your final answer.
- If the final result is too complex to compute, give the expression. e.g.  $C_{100}^{50}$  is acceptable.
- Problems are not necessarily ordered based on their difficulty levels.
- Always ask yourself what makes this problem a good practice?
- Read through the reference solution even if you can solve the problem for additional information which may help you to solve this type of problems.

## Legends

-  *Tips, additional information etc*
-  *Important theorem, conclusion to remember.*
-  *Addition questions for further study.*

## My Comments and Notes

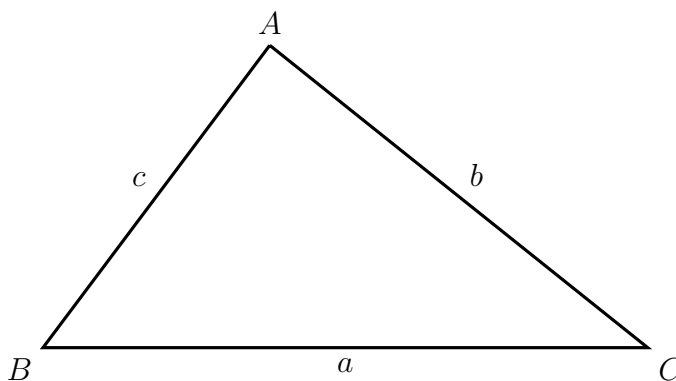
# Basic Trigonometry in Geometry



Unless otherwise noted, the following conventions will be followed in this practice:

In a given  $\triangle ABC$ :

- Uppercase letters  $A$ ,  $B$ , and  $C$  represent measurements of internal angles
- Lowercase letters  $a$ ,  $b$ , and  $c$  represent lengths of corresponding opposite sides



## Basic Trigonometry in Geometry



## Practice 1

Let  $S$  be the area of  $\triangle ABC$ , show that:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$$



## Practice 2

(Law of Sine) Show that the following relationship holds for any triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



## Practice 3

(Law of Cosine) Show that the following relationships hold for any triangle:

$$\begin{cases} a^2 = b^2 + c^2 - 2bc \cos A \\ b^2 = c^2 + a^2 - 2ca \cos B \\ c^2 = a^2 + b^2 - 2ab \cos C \end{cases}$$



## Practice 4

(Circumradius) Let  $R$  be the circumradius of  $\triangle ABC$ , show that

$$R = \frac{a}{2 \cdot \sin A} = \frac{b}{2 \cdot \sin B} = \frac{c}{2 \cdot \sin C}$$



## Basic Trigonometry in Geometry



## Practice 5

(Circumradius) Let  $S$  and  $R$  be the area and circumradius of  $\triangle ABC$ , respectively, show that

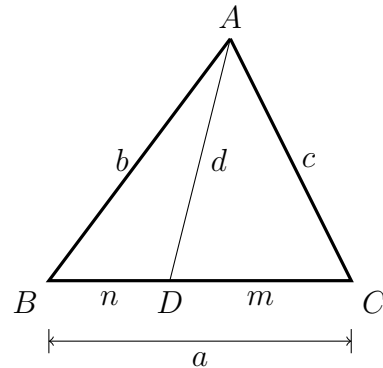
$$S = 2R^2 \sin A \sin B \sin C$$



## Practice 6

(Steward Theorem) Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show that the following relationship holds

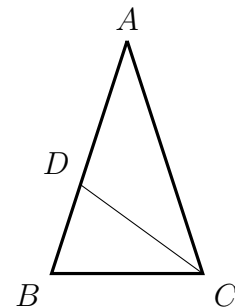
$$b^2m + c^2n = a(d^2 + mn)$$



## Practice 7

( $\sin 18^\circ$ ) Utilizing the graph on the right to compute the value of  $\sin 18^\circ$ .

$AB = AC$ ,  $\angle A = 36^\circ$ ,  $CD$  bisects  $\angle ACB$



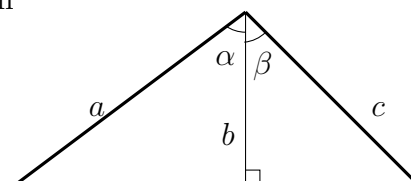
# Basic Trigonometry in Geometry



## Practice 8

(sum of sine) Utilizing the graph on the right to derive the sum of sine formula:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$



## Practice 9

(sin 15°) Show that  $\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ . Can you solve this problem using more than one method?

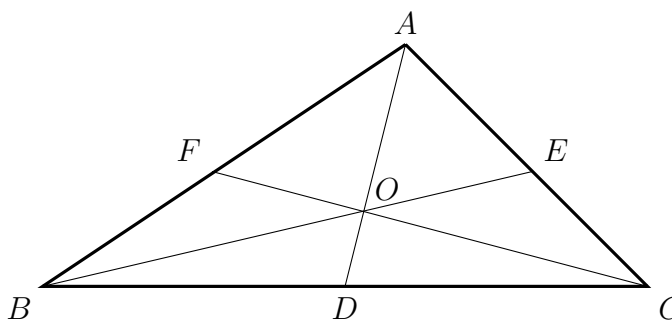


## Practice 10

(Trigonometry Form of Ceva's Theorem)

As shown on the right, show that:

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin ACF}{\sin FCB} \cdot \frac{\sin CBE}{\sin EBA} = 1$$



---

# Reference Solutions

---

## Basic Trigonometry in Geometry



## Practice 1

Let  $S$  be the area of  $\triangle ABC$ , show that:

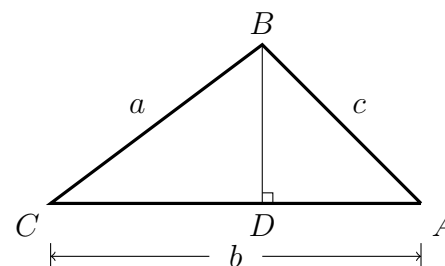
$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$$



Let's prove  $S = \frac{1}{2} \cdot ab \sin C$  here. The other two relationships can be proved in a similar way.

Draw an altitude from  $B$  and let its foot on  $AC$  be  $D$ . Then we have (note that  $\overline{BC} = a$  and  $\overline{AC} = b$ )

$$S = \frac{1}{2} \cdot \overline{AC} \cdot \overline{BD} = \frac{1}{2} \cdot \overline{AC} \cdot (\overline{BC} \cdot \sin C) = \frac{1}{2} \cdot ab \sin C$$



## Practice 2

**(Law of Sine)** Show that the following relationship holds for any triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



From the previous practice, we know:  $S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$ , or

$$bc \sin A = ca \sin B = ab \sin C$$

Dividing every term with  $abc$  which is non-zero leading to the conclusion:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



## Basic Trigonometry in Geometry



## Practice 3

(Law of Cosine) Show that the following relationships hold for any triangle:

$$\begin{cases} a^2 = b^2 + c^2 - 2bc \cos A \\ b^2 = c^2 + a^2 - 2ca \cos B \\ c^2 = a^2 + b^2 - 2ab \cos C \end{cases}$$

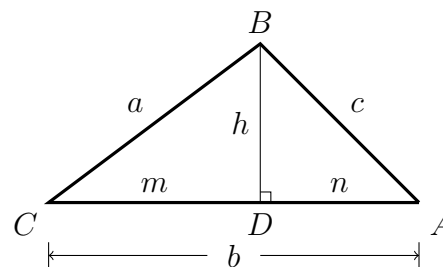


Let's prove the first relationship here. The other two can be proved similarly. There are several different proofs. Here we present two of them.

Proof 1

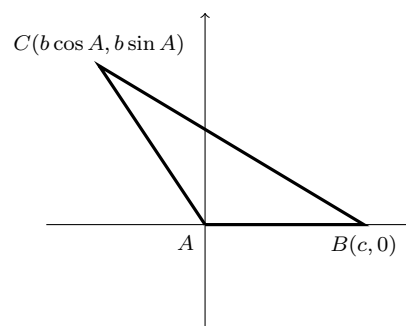
Draw an altitude from  $B$  and let its foot on  $AC$  be  $D$ . Note that  $\overline{BC} = a$ ,  $\overline{AC} = b$ , and  $\overline{AB} = c$ , we have:

$$\begin{aligned} a^2 &= h^2 + m^2 = (c^2 - n^2) + (b - n)^2 \\ &= c^2 - n^2 + b^2 - 2bn + n^2 \\ &= b^2 + c^2 - 2b \cdot n \\ &= b^2 + c^2 - 2bc \cdot \sin A \end{aligned}$$

Proof 2

Let's put  $\triangle ABC$  on a coordinate plane such that  $A$  is the origin and  $B$  is on the  $x$ -axis. It is then easy to see  $B$ 's coordinate is  $(c, 0)$  and  $C$ 's coordinate is  $(b \cos A, b \sin A)$ . Hence, by the distance formula, we have:

$$\begin{aligned} a^2 &= (b \cos A - c)^2 + (b \sin A - 0)^2 \\ &= b^2 \cos^2 A - 2bc \cdot \cos A + c^2 + b^2 \sin^2 A \\ &= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc \cdot \cos A \\ &= b^2 + c^2 - 2bc \cdot \cos A \end{aligned}$$



*Tip: The combination of trigonometry and coordinate system provides a powerful way to transform a geometry problem to a straightforward computation.*

## Basic Trigonometry in Geometry



## Practice 4

(Circumradius) Let  $R$  be the circumradius of  $\triangle ABC$ , show that

$$R = \frac{a}{2 \cdot \sin A} = \frac{b}{2 \cdot \sin B} = \frac{c}{2 \cdot \sin C}$$



Let's prove  $R = \frac{a}{2 \cdot \sin A}$  here. The other two relationships can be proved in a similar way.

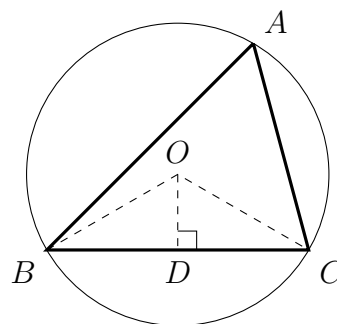
Let  $O$  be the circumcenter of  $\triangle ABC$ . Connect  $OB$ ,  $OC$ , and  $OD$ , where  $D$  is the middle point of  $BC$ .

$O$  is the circumcenter  $\implies OD \perp BC$  and  $OD$  bisects  $\angle BOC$ .

Meanwhile  $\angle BOC = 2\angle A \implies \angle BOD = \angle A$

Now consider right  $\triangle BOD$ , we have  $OB = R$ ,  $BD = \frac{a}{2}$ , and  $\angle BOD = \angle A$ . Therefore

$$BD = OB \cdot \sin \angle BOD \implies \frac{a}{2} = R \sin A \implies R = \frac{a}{2 \cdot \sin A}$$



## Practice 5

(Circumradius) Let  $S$  and  $R$  be the area and circumradius of  $\triangle ABC$ , respectively, show that

$$S = 2R^2 \sin A \sin B \sin C$$



By the previous practice, we know  $R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B}$ . Hence:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot (2R \sin A)(2R \sin B) \sin C = 2R^2 \sin A \sin B \sin C$$

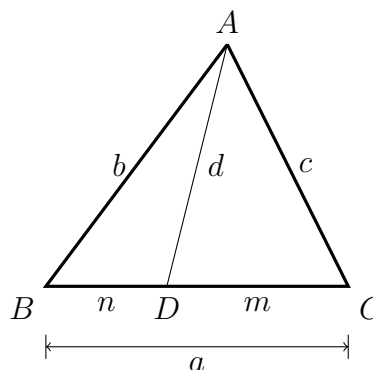
## Basic Trigonometry in Geometry



## Practice 6

**(Steward Theorem)** Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show that the following relationship holds

$$b^2m + c^2n = a(d^2 + mn)$$



By the Law of Cosine, we have:

$$\begin{cases} b^2 = n^2 + d^2 - 2nd \cdot \cos \angle BDA \\ c^2 = m^2 + d^2 - 2md \cdot \cos \angle CDA \end{cases}$$

Multiplying both sides of the 1<sup>st</sup> equation by  $m$ , both sides of the 2<sup>nd</sup> by  $n$ :

$$\begin{cases} b^2m = n^2m + d^2m - 2mnd \cdot \cos \angle BDA \\ c^2n = m^2n + d^2n - 2mnd \cdot \cos \angle CDA \end{cases}$$

Note that  $\angle BDA + \angle CDA = \pi \implies \cos \angle BDA + \cos \angle CDA = 0$ . Adding these two equations above give us:

$$\begin{aligned} b^2m + c^2n &= (n^2m + m^2n) + (d^2m + d^2n) \\ b^2m + c^2n &= (n + m)mn + d^2(m + n) \\ b^2m + c^2n &= (n + m)(mn + d^2) \\ b^2m + c^2n &= a(d^2 + mn) \end{aligned}$$

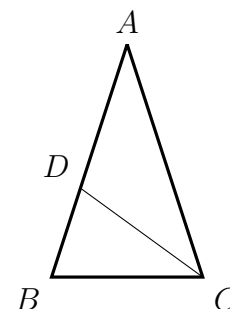
## Basic Trigonometry in Geometry



## Practice 7

( $\sin 18^\circ$ ) Utilizing the graph on the right to compute the value of  $\sin 18^\circ$ .

$AB = AC$ ,  $\angle A = 36^\circ$ ,  $CD$  bisects  $\angle ACB$

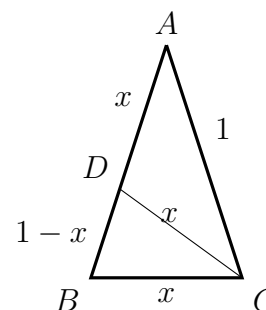


It is easy to check that  $\angle DCA = 36^\circ = \angle A$ , and  $\angle BDC = 72^\circ = \angle B$ .

Hence we have  $AD = DC = CB$ . Without loss of generality, let  $AC = AB = 1$ ,  $AD = DC = CB = x$ , and  $DB = 1 - x$ .

By the angle bisector theorem, we have

$$\frac{AC}{AD} = \frac{CB}{BD} \implies \frac{1}{x} = \frac{x}{1-x}$$



Solving the above equation and discarding the negative value give us  $x = \frac{\sqrt{5} - 1}{2}$ .

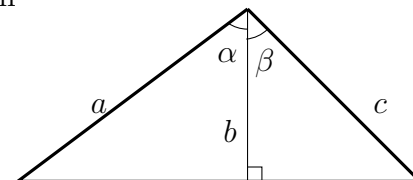
Because  $\angle A = 36^\circ$  and  $AB = AC$ , we have

$$\sin 18^\circ = \frac{x}{1} = \frac{\sqrt{5} - 1}{4}$$

## Practice 8

(**sum of sine**) Utilizing the graph on the right to derive the sum of sine formula:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$



We can employ the area method here. Clearly the area of the bigger triangle equals the sum of

## Basic Trigonometry in Geometry



those of the two smaller ones. Therefore, we have:

$$\frac{1}{2}ac \cdot \sin(\alpha + \beta) = \frac{1}{2}ab \cdot \sin \alpha + \frac{1}{2}bc \cdot \sin \beta$$

$$\sin(\alpha + \beta) = \frac{b}{c} \cdot \sin \alpha + \frac{b}{a} \cdot \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

## Practice 9

(sin 15°) Show that  $\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ . Can you solve this problem using more than one method?



**i** *Tip: This problem can be solved using the angle bisector theorem, just as we did to compute  $\sin 18^\circ$  earlier. Meanwhile, it can also be solved using the Sum of Sine formula we just derived.*

**i** *Tip: We will show the 2<sup>nd</sup> approach. It involves some techniques to simplify computation. You are encouraged to use the angle bisector theorem to solve this problem yourself.*

Setting  $\alpha = \beta$  in the sum of sine formula leads to the double angle formula:

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

Further setting  $\alpha = 15^\circ$  yields:

$$\sin 30^\circ = 2 \cdot \sin 15^\circ \cos 15^\circ$$

Let  $x = \sin 15^\circ$ , we have

$$\frac{1}{2} = 2x\sqrt{1-x^2}$$

$$1 = 4x\sqrt{1-x^2}$$

$$1 = 16x^2(1-x^2)$$

$$16x^4 - 16x^2 + 1 = 0$$

$$(4x^2 - 2)^2 = 3$$

$$x^2 = \frac{2 \pm \sqrt{3}}{4}$$

## Basic Trigonometry in Geometry



Because  $0 < \sin 15^\circ < \sin 30^\circ = \frac{1}{2} \implies x^2 < \frac{1}{4}$ , it must hold that

$$x^2 = \frac{2 - \sqrt{3}}{4} \implies x = \frac{\sqrt{2 - \sqrt{3}}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

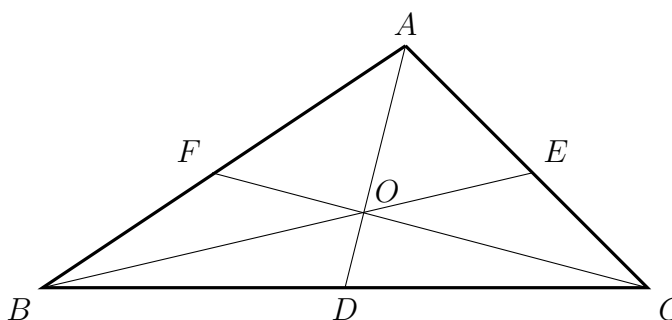
**i** *Tip: The technique used to simplify nested radical expression in the last step is discussed in the Power Calculation lecture and practice.*

## Practice 10

**(Trigonometry Form of Ceva's Theorem)**

As shown on the right, show that:

$$\frac{\sin \angle BAD}{\sin \angle DAC} \cdot \frac{\sin \angle ACF}{\sin \angle FCB} \cdot \frac{\sin \angle CBE}{\sin \angle EBA} = 1$$



**i** *Tip: When cevians and ratios are involved, the area method is always a good candidate to consider.*

We have the following relationship:

$$\frac{S_{\triangle ABO}}{S_{\triangle AOC}} = \frac{\frac{1}{2} \cdot AB \cdot AO \cdot \sin \angle BAD}{\frac{1}{2} \cdot AO \cdot AC \cdot \sin \angle DAC} = \frac{AB \cdot \sin \angle BAD}{AC \cdot \sin \angle DAC} \implies \frac{\sin \angle BAD}{\sin \angle DAC} = \frac{S_{\triangle ABO} \cdot AC}{S_{\triangle AOC} \cdot AB}$$

Similarly, we have:

$$\frac{\sin \angle ACF}{\sin \angle FCB} = \frac{S_{\triangle ACO} \cdot BC}{S_{\triangle OCB} \cdot AC}$$

$$\frac{\sin \angle CBE}{\sin \angle EBA} = \frac{S_{\triangle CBO} \cdot AB}{S_{\triangle OBA} \cdot BC}$$

Multiplying these three equations gives us:

$$\frac{\sin \angle BAD}{\sin \angle DAC} \cdot \frac{\sin \angle ACF}{\sin \angle FCB} \cdot \frac{\sin \angle CBE}{\sin \angle EBA} = \frac{S_{\triangle ABO} \cdot AC}{S_{\triangle AOC} \cdot AB} \cdot \frac{S_{\triangle ACO} \cdot BC}{S_{\triangle OCB} \cdot AC} \cdot \frac{S_{\triangle CBO} \cdot AB}{S_{\triangle OBA} \cdot BC} = 1$$