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# Geometry

# Basic Trigonometry in Geometry



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# Basic Trigonometry in Geometry



### Instructions

- Write down and submit intermediate steps along with your final answer.
- If the final result is too complex to compute, give the expression. e.g.  $C_{100}^{50}$  is acceptable.
- Problems are not necessarily ordered based on their difficulty levels.
- Always ask yourself what makes this problem a good practice?
- Read through the reference solution even if you can solve the problem for additional information which may help you to solve this type of problems.

### Legends

- Tips, additional information etc
- ✓ Important theorem, conclusion to remember.
- Addition questions for further study.

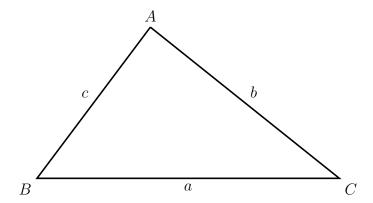
### My Comments and Notes

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Unless otherwise noted, the following conventions will be followed in this practice: In a given  $\triangle ABC$ :

- $\bullet$  Uppercase letters A, B, and C represent measurements of internal angles
- $\bullet$  Lowercase letters a, b, and c represent lengths of corresponding opposite sides



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### Practice 1

Let S be the area of  $\triangle ABC$ , show that:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$$



### Practice 2

(Law of Sine) Show that the following relationship holds for any triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



### Practice 3

(Law of Cosine) Show that the following relationships hold for any triangle:

$$\begin{cases} a^2 = b^2 + c^2 - 2bc \cos A \\ b^2 = c^2 + a^2 - 2ca \cos B \\ c^2 = a^2 + b^2 - 2ab \cos C \end{cases}$$



### Practice 4

(Circumradius) Let R be the circumradius of  $\triangle ABC$ , show that

$$R = \frac{a}{2 \cdot \sin A} = \frac{b}{2 \cdot \sin B} = \frac{c}{2 \cdot \sin C}$$



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### Practice 5

(Circumradius) Let S and R be the area and circumradius of  $\triangle ABC$ , respectively, show that

$$S = 2R^2 \sin A \sin B \sin C$$

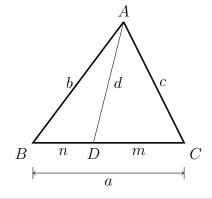


#### Practice 6

(Steward Theorem) Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show that the following relationship holds

$$b^2m + c^2n = a(d^2 + mn)$$



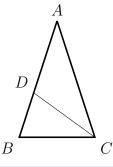


### Practice 7

( $\sin 18^{\circ}$ ) Utilizing the graph on the right to compute the value of  $\sin 18^{\circ}$ .

$$AB = AC$$
,  $\angle A = 36^{\circ}$ ,  $CD$  bisects  $\angle ACB$ 





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### Practice 8

(sum of sine) Utilizing the graph on the right to derive the sum of sine formula:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$



## Practice 9

(sin 15°) Show that  $\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ . Can you solve this problem using more than one method?



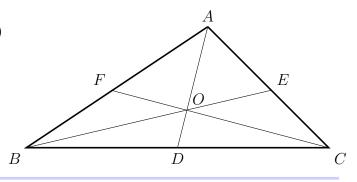
#### Practice 10

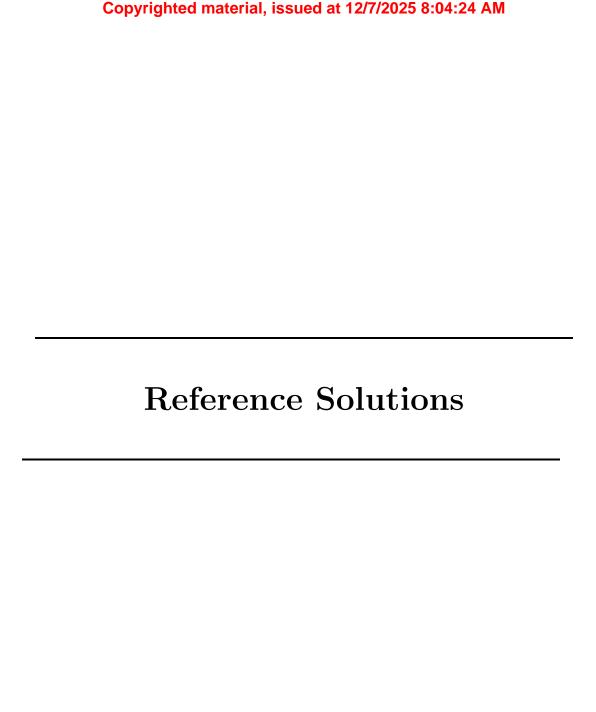
(Trigonometry Form of Civa's Theorem)

As shown on the right, show that:

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin ACF}{\sin FCB} \cdot \frac{\sin CBE}{\sin EBA} = 1$$







# Basic Trigonometry in Geometry



#### Practice 1

Let S be the area of  $\triangle ABC$ , show that:

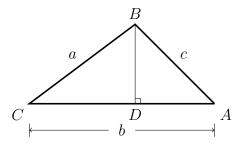
$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$$



Let's prove  $S = \frac{1}{2} \cdot ab \sin C$  here. The other two relationships can be proved in a similar way.

Draw an altitude from B and let its foot on AC be D. Then we have (note that  $\overline{BC}=a$  and  $\overline{AC}=b$ )

$$S = \frac{1}{2} \cdot \overline{AC} \cdot \overline{BD} = \frac{1}{2} \cdot \overline{AC} \cdot (\overline{BC} \cdot \sin C) = \frac{1}{2} \cdot ab \sin C$$



### Practice 2

(Law of Sine) Show that the following relationship holds for any triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



From the previous practice, we know:  $S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot bc \sin A = \frac{1}{2} \cdot ca \sin B$ , or

$$bc\sin A = ca\sin B = ab\sin C$$

Dividing every term with abc which is non-zero leading to the conclusion:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

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#### Practice 3

(Law of Cosine) Show that the following relationships hold for any triangle:

$$\begin{cases} a^2 = b^2 + c^2 - 2bc \cos A \\ b^2 = c^2 + a^2 - 2ca \cos B \\ c^2 = a^2 + b^2 - 2ab \cos C \end{cases}$$



Let's prove the first relationship here. The other two can be proved similarly. There are several different proofs. Here we present two of them.

#### Proof 1

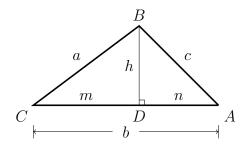
Draw an altitude from B and let its foot on AC be D. Note that  $\overline{BC} = a$ ,  $\overline{AC} = b$ , and  $\overline{AB} = c$ , we have:

$$a^{2} = h^{2} + m^{2} = (c^{2} - n^{2}) + (b - n)^{2}$$

$$= c^{2} - n^{2} + b^{2} - 2bn + n^{2}$$

$$= b^{2} + c^{2} - 2b \cdot n$$

$$= b^{2} + c^{2} - 2bc \cdot \sin A$$



#### Proof 2

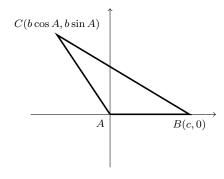
Let's put  $\triangle ABC$  on a coordinate plane such that A is the origin and B is on the x-axis. It is then easy to see B's coordinate is (c,0) and C's coordinate is  $(b\cos A, b\sin A)$ . Hence, by the distance formula, we have:

$$a^{2} = (b\cos A - c)^{2} + (b\sin A - 0)^{2}$$

$$= b^{2}\cos^{2} A - 2bc \cdot \cos A + c^{2} + b^{2}\sin^{2} A$$

$$= b^{2}(\cos^{2} A + \sin^{2} A) + c^{2} - 2bc \cdot \cos A$$

$$= b^{2} + c^{2} - 2bc \cdot \cos A$$



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Tip: The combination of trigonometry and coordinate system provides a powerful way to transform a geometry problem to a straightforward computation.

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A

B

#### Practice 4

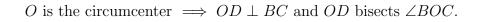
(Circumradius) Let R be the circumradius of  $\triangle ABC$ , show that

$$R = \frac{a}{2 \cdot \sin A} = \frac{b}{2 \cdot \sin B} = \frac{c}{2 \cdot \sin C}$$

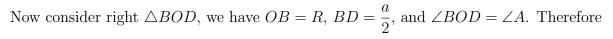


Let's prove  $R = \frac{a}{2 \cdot \sin A}$  here. The other two relationships can be proved in a similar way.

Let O be the circumcircle of  $\triangle ABC$ . Connect OB, OC, and OD, where D is the middle point of BC.







$$BD = OB \cdot \angle BOD \implies \frac{a}{2} = R \sin A \implies R = \frac{a}{2 \cdot \sin A}$$

### Practice 5

(Circumradius) Let S and R be the area and circumradius of  $\triangle ABC$ , respectively, show that

$$S = 2R^2 \sin A \sin B \sin C$$



By the previous practice, we know  $R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B}$ . Hence:

$$S = \frac{1}{2} \cdot ab \sin C = \frac{1}{2} \cdot (2R \sin A)(2R \sin C) = 2R^2 \sin A \sin B \sin C$$

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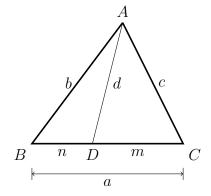


#### Practice 6

(Steward Theorem) Given a triangle as shown on the right where each letter represents the length of a corresponding segment, show that the following relationship holds

$$b^2m + c^2n = a(d^2 + mn)$$





By the Law of Cosine, we have:

$$\begin{cases} b^2 = n^2 + d^2 - 2nd \cdot \cos \angle BDA \\ c^2 = m^2 + d^2 - 2md \cdot \cos \angle CDA \end{cases}$$

Multiplying both sides of the  $1^{st}$  equation by m, both sides of the  $2^{nd}$  by n:

$$\begin{cases} b^2m = n^2m + d^2m - 2mnd \cdot \cos \angle BDA \\ c^2n = m^2n + d^2n - 2mnd \cdot \cos \angle CDA \end{cases}$$

Note that  $\angle BDA + \angle CDA = \pi \implies \cos \angle BDA + \cos \angle CDA = 0$ . Adding these two equations above give us:

$$b^{2}m + c^{2}n = (n^{2}m + m^{2}n) + (d^{2}m + d^{2})n$$

$$b^{2}m + c^{2}n = (n + m)mn + d^{2}(m + n)$$

$$b^{2}m + c^{2}n = (n + m)(mn + d^{2})$$

$$b^{2}m + c^{2}n = a(d^{2} + mn)$$

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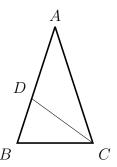


### Practice 7

( $\sin 18^{\circ}$ ) Utilizing the graph on the right to compute the value of  $\sin 18^{\circ}$ .

$$AB = AC$$
,  $\angle A = 36^{\circ}$ ,  $CD$  bisects  $\angle ACB$ 



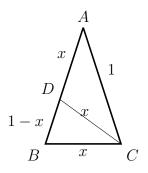


It is easy to check that  $\angle DCA = 36^{\circ} = \angle A$ , and  $\angle BDC = 72^{\circ} = \angle B$ .

Hence we have AD = DC = CB. Without loss of generality, let AC = AB = 1, AD = DC = CB = x, and DB = 1 - x.

By the angle bisector theorem, we have

$$\frac{AC}{AD} = \frac{CB}{BD} \implies \frac{1}{x} = \frac{x}{1-x}$$



Solving the above equation and discarding the negative value give us  $x = \frac{\sqrt{5}-1}{2}$ .

Because  $\angle A = 36^{\circ}$  and AB = AC, we have

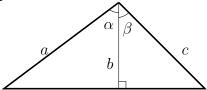
$$\sin 18^{\circ} = \frac{\frac{x}{2}}{1} = \frac{\sqrt{5} - 1}{4}$$

### Practice 8

(sum of sine) Utilizing the graph on the right to derive the sum of sine formula:

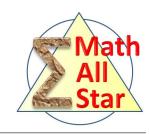
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$





We can employ the area method here. Clearly the area of the bigger triangle equals the sum of

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those of the two smaller ones. Therefore, we have:

$$\frac{1}{2}ac \cdot \sin(\alpha + \beta) = \frac{1}{2}ab \cdot \sin\alpha + \frac{1}{2}bc \cdot \sin\beta$$
$$\sin(\alpha + \beta) = \frac{b}{c} \cdot \sin\alpha + \frac{b}{a} \cdot \sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

#### Practice 9

(sin 15°) Show that  $\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ . Can you solve this problem using more than one method?



- Tip: This problem can be solved using the angle bisector theorem, just as we did to compute sin 18° earlier. Meanwhile, it can also be solved using the Sum of Sine formula we just derived.
- Tip: We will show the 2<sup>nd</sup> approach. It involves some techniques to simplify computation. You are encouraged to use the angle bisector theorem to solve this problem yourself.

Setting  $\alpha = \beta$  in the sum of sine formula leads to the double angle formula:

$$\sin 2\alpha = 2\sin \alpha \cos \alpha$$

Further setting  $\alpha = 15^{\circ}$  yields:

$$\sin 30^{\circ} = 2 \cdot \sin 15^{\circ} \cos 15^{\circ}$$

Let  $x = \sin 15^{\circ}$ , we have

$$\frac{1}{2} = 2x\sqrt{1 - x^2}$$

$$1 = 4x\sqrt{1 - x^2}$$

$$1 = 16x^2(1 - x^2)$$

$$16x^4 - 16x^2 + 1 = 0$$

$$(4x^2 - 2)^2 = 3$$

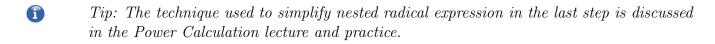
$$x^2 = \frac{2 \pm \sqrt{3}}{4}$$

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Because  $0 < \sin 15^{\circ} < \sin 30^{\circ} = \frac{1}{2} \implies x^2 < \frac{1}{4}$ , it must hold that

$$x^{2} = \frac{2 - \sqrt{3}}{4} \implies x = \frac{\sqrt{2 - \sqrt{3}}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$



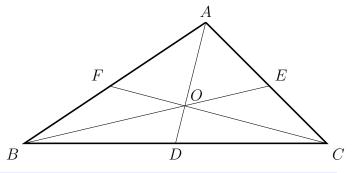
#### Practice 10

(Trigonometry Form of Civa's Theorem)

As shown on the right, show that:

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin ACF}{\sin FCB} \cdot \frac{\sin CBE}{\sin EBA} = 1$$





Tip: When cevians and ratios are involved, the area method is always a good candidate to consider.

We have the following relationship:

$$\frac{S_{\triangle ABO}}{S_{\triangle AOC}} = \frac{\frac{1}{2} \cdot AB \cdot AO \cdot \sin \triangle BAD}{\frac{1}{2} \cdot AO \cdot AC \cdot \sin \triangle DAC} = \frac{AB \cdot \sin \angle BAD}{AC \cdot \sin \angle DAC} \implies \frac{\sin \angle BAD}{\sin \angle DAC} = \frac{S_{\triangle ABO} \cdot AC}{S_{\triangle OAC} \cdot AB}$$

Similarly, we have:

$$\frac{\sin \angle ACF}{\sin \angle FCB} = \frac{S_{\triangle ACO} \cdot BC}{S_{\triangle OCB} \cdot AC}$$
$$\frac{\sin \angle CBE}{\sin \angle EBA} = \frac{S_{\triangle CBO} \cdot AB}{S_{\triangle OBA} \cdot BC}$$

Multiplying these three equations gives us:

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin ACF}{\sin FCB} \cdot \frac{\sin CBE}{\sin EBA} = \frac{S_{\triangle ABO} \cdot AC}{S_{\triangle OAC} \cdot AB} \cdot \frac{S_{\triangle ACO} \cdot BC}{S_{\triangle OCB} \cdot AC} \cdot \frac{S_{\triangle CBO} \cdot AB}{S_{\triangle OBA} \cdot BC} = 1$$