Geometry

Triangle Basic

Math All Star

Math for Gifted Students

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Geometry

Triangle Basic

Instructions

- *Write down and submit intermediate steps along with your final answer.*
- *If the final result is too complex to compute, give the expression. e.g.* C_{100}^{50} *is acceptable.*
- *Problems are not necessarily ordered based on their difficulty levels.*
- *Always ask yourself what makes this problem a good practice?*
- *Read through the reference solution even if you can solve the problem for additional information which may help you to solve this type of problems.*

Legends

ℹ *Tips, additional information etc*

✔ *Important theorem, conclusion to remember.*

? *Addition questions for further study.*

My Comments and Notes

Triangle Basic

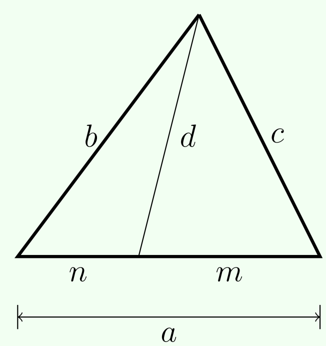
Reference

You may need to use the following theorem(s) and conclusion(s) in this practice:

Theorem 0.0.1 Stewart Theorem

Given a triangle as shown on the right where each letter represents the length of a corresponding segment, then

$$b^2m + c^2n = a(d^2 + mn)$$



Triangle Basic

Practice 1

In $\triangle ABC$, $\angle C$ is a right angle. Let D be the middle point of AB. Show that $CD = \frac{1}{2} \cdot AB$.

Practice 2

In $\triangle ABC$, $\angle C$ is a right angle. Let D be the foot of the altitude drawn from C on AB. Show that:

(i) $CD = \dfrac{AC \cdot BC}{AB} = \dfrac{AC \cdot BC}{\sqrt{AC^2 + BC^2}}$

(ii) $CD^2 = AD \cdot BD$

(iii) $AC^2 = AD \cdot AB$ and $BC^2 = BD \cdot AB$

Practice 3

Let a, b, and c be the lengths of two legs and the hypotenuse of a right triangle respectively. Let r be the radius of this triangle's inscribed circle. Show that

$$r = \frac{a + b - c}{2}$$

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Triangle Basic

Practice 4

(Apollonius Theorem) Given $\triangle ABC$ and median AD, show that

$$AB^2 + AC^2 = 2(AD^2 + BD \cdot CD)$$

Practice 5

(Extended Pythagorean Theorem) Let $\triangle ABC$ be a right triangle where $\angle C = 90°$. If point D is on side BC or its extension, show that

$$AB^2 = DB^2 + DA^2 \pm 2 \cdot DB \cdot DC$$

If D is on BC, then the 3^{rd} term above takes a positive coefficient. If D is on BC's extension, the 3^{rd} term takes a negative coefficient.

Practice 6

(Angle Bisector Theorem) In $\triangle ABC$, let D be the foot of $\angle A$ bisector on BC. Show that

$$\frac{AB}{BD} = \frac{AC}{CD}$$

Practice 7

Let $\triangle ABC$ be an isosceles triangle where $AB = AC$. Show that for any point P on the base BC or its extension, the following relationship holds:

$$AP^2 = AB \cdot AC \pm BP \cdot PC$$

(Ref 2903)

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Triangle Basic

Practice 8

In equiangular hexagon $ABCDEF$, if $AB + BC = 11$ and $FA - CD = 3$, compute $BC + DE$.
(Ref 2904: 1994 China Beijing)

Practice 9

(Civa's Theorem) Given a triangle ABC, let the lines AO, BO and CO be drawn from the vertices to a common point O to meet opposite sides at D, E and F respectively. (The segments AD, BE, and CF are known as cevians.) Then, show the following equation holds:

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

Practice 10

(Menelaus' theorem) Given a triangle ABC, and a transversal line that crosses BC, AC and AB (or their extended segment) at points D, E and F respectively, with D, E, and F distinct from A, B and C, show that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$$

(Note: the negative 1 assumes the length of a segment is signed, e.g. $AB = -BA$. If this is confusing to you, simply assume the length is always positive, and then to prove the product equals 1.)

Reference Solutions

Triangle Basic

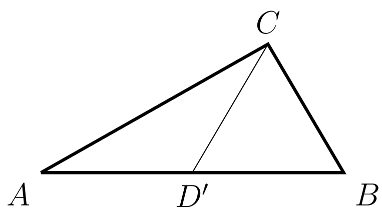
Practice 1

In $\triangle ABC$, $\angle C$ is a right angle. Let D be the middle point of AB. Show that $CD = \frac{1}{2} \cdot AB$.

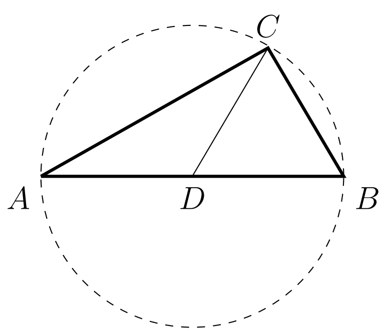
There are several different ways to prove this:

Method 1: Find a point D' on AB such that $AD' = CD'$. This means $\angle A = \angle ACD'$. It follows that $\angle BCD' = 90° - \angle ACD' = 90° - \angle A = \angle B$. Hence we find $\triangle CD'B$ is isosceles where $BD' = CD'$, or

$$D'C = AD' = D'B \implies D' \text{ is the middle point of } AB \text{ , i.e. } D \text{ , and } CD = \frac{1}{2} \cdot AB$$



Method 2: Draw a circle which is centered at D with a radius of $AD = DB$. Because $\angle C$ is a right angle, point C must locate on this circle. Hence CD must equal to this circle's radius which is half of AB.



Method 3: By the Steward theorem, we have

$$CA^2 \cdot DB + CB^2 \cdot AD = AB \cdot (CD^2 + AD \cdot DB)$$

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Note that $AD = BD = \frac{1}{2} \cdot AB$ and $CA^2 + CB^2 = AB^2$ (Pythagorean theorem):

$$\begin{aligned} CA^2 \cdot DB + CB^2 \cdot AD &= AB \cdot (CD^2 + AD \cdot DB) \\ CA^2 + CB^2 &= 2 \cdot (CD^2 + AD \cdot DB) \\ AB^2 &= 2 \cdot CD^2 + 2 \cdot (\frac{1}{2} \cdot AB)(\frac{1}{2} \cdot AB) \\ AB^2 &= 2 \cdot CD^2 + \frac{1}{2} \cdot AB^2 \\ AB^2 &= 4 \cdot CD^2 \\ AB &= 2 \cdot CD \end{aligned}$$

Practice 2

In $\triangle ABC$, $\angle C$ is a right angle. Let D be the foot of the altitude drawn from C on AB. Show that:

(i) $CD = \frac{AC \cdot BC}{AB} = \frac{AC \cdot BC}{\sqrt{AC^2 + BC^2}}$

(ii) $CD^2 = AD \cdot BD$

(iii) $AC^2 = AD \cdot AB$ and $BC^2 = BD \cdot AB$

It is always helpful to draw a correct diagram when solving a geometry problem.

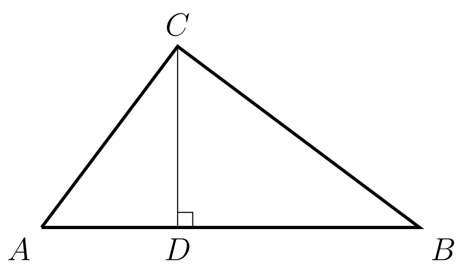

Figure 1:

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(i) This can be proved using the area method. Let S be the area of $\triangle ABC$. We use two different ways to compute S:

$$\text{method 1:} \quad AC \perp BC \implies S = \frac{1}{2} \cdot AC \cdot BC$$

$$\text{method 2:} \quad AB \perp CD \implies S = \frac{1}{2} \cdot AB \cdot CD$$

Therefore it must hold that $\frac{1}{2} \cdot AC \cdot BC = \frac{1}{2} \cdot AB \cdot CD$, or

$$CD = \frac{AC \cdot BC}{AB} = \frac{AC \cdot BC}{\sqrt{AC^2 + BC^2}}$$

The 2^{nd} equality above is based on the Pythagorean theorem.

Tip: This presents one way to compute length of the altitude CD using the side lengths.

(ii) This can be proved using similar triangles.

Because $\angle ACD = 90° - \angle DCB = \angle CBD$, and $90° = \angle ADC = \angle CDB$, we have

$$\triangle ADC \sim \triangle CDB$$

Therefore

$$\frac{AD}{CD} = \frac{CD}{DB} \qquad \text{or} \qquad CD^2 = AD \cdot BD$$

Tip: This presents another way to compute the altitude from the right angle.

(iii) This can be proved using similar triangles.

Because $\angle ACD = 90° - \angle DCB = \angle CBD$, and $90° = \angle ADC = \angle ACB$, we have

$$\triangle ADC \sim \triangle ACB$$

Therefore

$$\frac{AC}{AD} = \frac{AB}{AC} \qquad \text{or} \qquad AC^2 = AD \cdot AB$$

The other formula can be proved in a similar way.

Tip: We can view AD as the projection of the side AC on the hypotenuse AB. This formula relates lengths of the side, its projection, and the hypotenuse.

Quiz: How many similar triangles are there in Figure 1?

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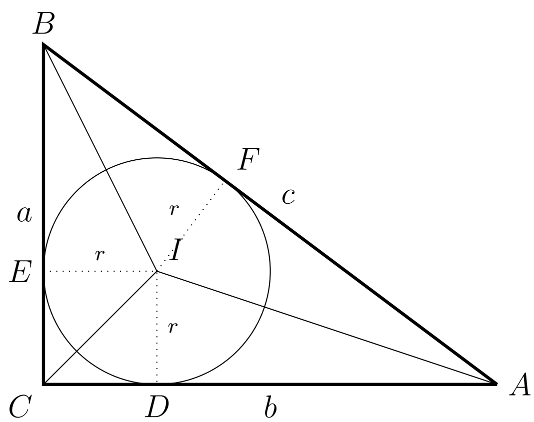
Practice 3

Let a, b, and c be the lengths of two legs and the hypotenuse of a right triangle respectively. Let r be the radius of this triangle's inscribed circle. Show that

$$r = \frac{a+b-c}{2}$$

This can be proved using the area method (please check out the area method practice). Here we provide another solution.



Let D, E, and F be the points of tangency, respectively. Then we have $AD = AF$, $BF = BE$, and $CD = CE$. Because $\angle C = 90°$, it is easy to see that $CDIE$ is a square. Hence:

$$2r = CE + CD = (a - BE) + (b - AD) = a + b - (BE + AD) = a + b - (BF + AF) = a + b - c$$

or

$$r = \frac{a+b-c}{2}$$

Tip: If the given triangle is not a right triangle, we can still obtain the in-radius formula by employing the area method.

Triangle Basic

Practice 4

(Apollonius Theorem) Given $\triangle ABC$ and median AD, show that

$$AB^2 + AC^2 = 2(AD^2 + BD \cdot CD)$$

By the Stewart Theorem, we have

$$AB^2 \cdot CD + AC^2 \cdot BD = BC \cdot (AD^2 + BD \cdot CD)$$

Note that $BD = CD = \frac{1}{2} \cdot BC$, the above relationship can be simplified to:

$$AB^2 + AC^2 = 2(AD^2 + BC \cdot CD)$$

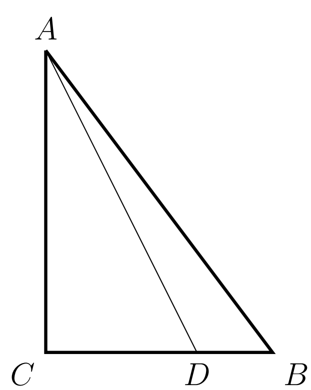
Practice 5

(Extended Pythagorean Theorem) Let $\triangle ABC$ be a right triangle where $\angle C = 90°$. If point D is on side BC or its extension, show that

$$AB^2 = DB^2 + DA^2 \pm 2 \cdot DB \cdot DC$$

If D is on BC, then the 3^{rd} term above takes a positive coefficient. If D is on BC's extension, the 3^{rd} term takes a negative coefficient.

Case 1: when D is on segment BC



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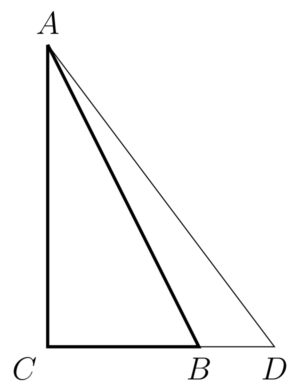
Apply the Pythagorean theorem on $\triangle ABC$ and $\triangle ADC$, respectively:

$$AB^2 = AC^2 + BC^2$$
$$DA^2 = AC^2 + DC^2$$

Canceling AC^2 leads to:

$$\begin{aligned} AB^2 &= DA^2 + BC^2 - DC^2 \\ &= DA^2 + (DB + DC)^2 - DC^2 \\ &= DB^2 + DA^2 + 2 \cdot DB \cdot DC \end{aligned}$$

Case 2: when D is on the extension of BC



Still apply the Pythagorean theorem on $\triangle ABC$ and $\triangle ADC$, respectively:

$$AB^2 = AC^2 + BC^2$$
$$DA^2 = AC^2 + DC^2$$

Canceling AC^2 leads to:

$$\begin{aligned} AB^2 &= DA^2 + BC^2 - DC^2 \\ &= DA^2 + (DC - DB)^2 - DC^2 \\ &= DB^2 + DA^2 - 2 \cdot DB \cdot DC \end{aligned}$$

? *Quiz: Draw a diagram when D is on the other side extension of BC, and provide a proof.*

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Practice 6

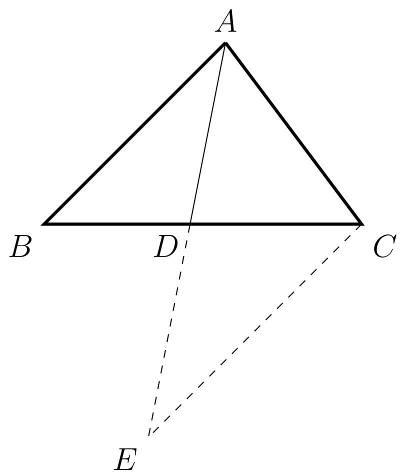
(Angle Bisector Theorem) In $\triangle ABC$, let D be the foot of $\angle A$ bisector on BC. Show that

$$\frac{AB}{BD} = \frac{AC}{CD}$$

Extending AD to point E such that $AC = CE$. It follows that $\angle CED = \angle CAD = \angle DAB$. Note that $\angle CDE = \angle BDA$, then $\triangle ABD \sim \triangle ECD$. Hence

$$\frac{AB}{BD} = \frac{CE}{CD}$$

By construction, we have $CE = AC$. Therefore the claim holds.



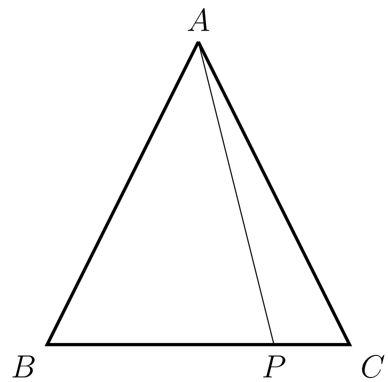
Practice 7

Let $\triangle ABC$ be an isosceles triangle where $AB = AC$. Show that for any point P on the base BC or its extension, the following relationship holds:

$$AP^2 = AB \cdot AC \pm BP \cdot PC$$

(Ref 2903)

Case 1: when P is on the segment BC.



By the Steward theorem, we have:

$$AB^2 \cdot PC + AC^2 \cdot BP = BC \cdot (AP^2 + BP \cdot PC)$$

Note $AB = AC$ and $(PC + BP) = BC$, we have

$$\begin{aligned} AB^2 \cdot PC + AC^2 \cdot BP &= BC \cdot (AP^2 + BP \cdot PC) \\ AB^2 \cdot (PC + BP) &= BC \cdot (AP^2 + BP \cdot PC) \\ AB^2 &= AP^2 + BP \cdot PC \\ AP^2 &= AB^2 - BP \cdot PC \\ AP^2 &= AB \cdot AC - BP \cdot PC \end{aligned}$$

Quiz: Draw a diagram when P is on the extension of BC, and then prove the conclusion.

Practice 8

In equiangular hexagon $ABCDEF$, if $AB + BC = 11$ and $FA - CD = 3$, compute $BC + DE$.
(Ref 2904: 1994 China Beijing)

Extending sides AF, BC, and DE so that they intersect at X, Y, and Z, respectively.

Tip: When an equiangular hexagon is involved, try to construct an equilateral triangle from it.

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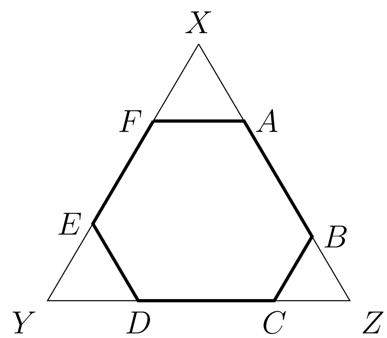

Because $ABCDEF$ is equiangular, every inner angle equals $120°$. It follows that the three corner triangles, XAB, CDZ, and FYE are all equilateral. Furthermore, the big triangle XYZ is also equilateral.

Now it is just a straightforward computation in order to solve the given problem:

$$\begin{aligned} BC + DE &= CZ + DY \\ &= YZ - DC \\ &= XZ - DC \\ &= (XA + AB + BZ) - DC \\ &= (FA + AB + BC) - DC \\ &= (FA - DC) + (AB + BC) \\ &= 3 + 11 \\ &= 14 \end{aligned}$$

Practice 9

(Civa's Theorem) Given a triangle ABC, let the lines AO, BO and CO be drawn from the vertices to a common point O to meet opposite sides at D, E and F respectively. (The segments AD, BE, and CF are known as cevians.) Then, show the following equation holds:

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

This problem can be solved using pure geometry approach. Here we present another approach which utilizes a physical concept: center of mass.

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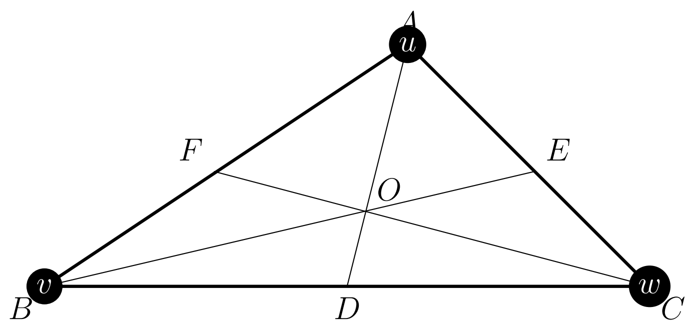
Tip: The center of mass method is a powerful tool to solve cevians related problems.

The objective is to place some proportional weights at the vertices A, B, and C, respectively, such that point O is the center of mass of these three weights. Then we can apply the balance condition to derive our results.



Let's place weights u, v, and w at point A, B, and C, respectively such that:

$$u \cdot AF = v \cdot BF \qquad \text{and} \qquad u \cdot AE = w \cdot CE$$

By the center of mass equation, we know F is the center of mass of the two weights u and v. Similarly, E is the center of mass of the two weights u and w.

Therefore, the intersection point of BE and CF, or point O, must be the center of mass of these three weights. It follows that D must be center of mass of weights v and w, or

$$v \cdot BD = w \cdot CD$$

Now, we have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{v}{u} \cdot \frac{w}{v} \cdot \frac{u}{w} = 1$$

Tip: To correctly remember the numerator and denominator of each of these three terms, we can start from any of the three vertices and write down all the segments in turn clockwise (or anti-clockwise).

Triangle Basic

Practice 10

(Menelaus' theorem) Given a triangle ABC, and a transversal line that crosses BC, AC and AB (or their extended segment) at points D, E and F respectively, with D, E, and F distinct from A, B and C, show that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$$

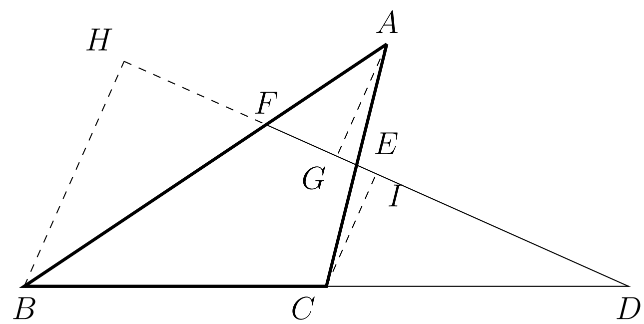
(Note: the negative 1 assumes the length of a segment is signed, e.g. $AB = -BA$. If this is confusing to you, simply assume the length is always positive, and then to prove the product equals 1.)

Tip: This theorem can also be proved using the center of mass method. When the desired center of mass lies outside the two weights, we can assume one weight is negative. Conceptually, a positive wight is equivalent to pushing down, and a negative weight is equivalent to pulling up.)

Quiz: Try to prove this theorem using the center of mass method.

Here, we are going to prove this theorem using a pure geometry approach.

Let's draw three perpendicular lines from vertices A, B, C towards the transversal line, and let their feet be G, H, and I, respectively.



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Therefore, we have

$$\triangle AFG \sim \triangle BFH \implies \frac{AF}{FB} = \frac{AG}{HB}$$
$$\triangle BDH \sim \triangle CDE \implies \frac{BD}{DC} = \frac{BH}{IC}$$
$$\triangle CIE \sim \triangle AGE \implies \frac{CE}{EA} = \frac{CI}{GA}$$

Multiplying both sides of the above three relationship gives us the desired result.

Tip: In this proof, the transversal line intersects two sides AB and AC, but intersects the 3^{rd} side BC on its extension. Note that the Menelaus theorem will still hold if the line DF intersects more than one side on their extensions.

Quiz: Draw a diagram when DF intersects all the three sides on their extensions, and then apply a similar method to prove it.

Battle Field

Selective problems from recent comptitions:

Problem 1: 2015 MathCounts State Team #7 (Ref 541)

Problem 2: 2014 AMC10A #14 (Ref 1293)

Problem 3: 2012 MathCounts State Sprint #21 (Ref 1923)

Problem 4: 2010AMC12A #17 (Ref 660)

Problem 5: 2009 AMC10A #10 (Ref 1544)

Problem 6: 2005 AMC10B #10 (Ref 1769)