MOD Basic

Math All Star Practice by Subject Series

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How to get the most out of practice

The most important tip is to learn before practice. This approach is what students do in schools. It also applies to competition math. Systematic learning will help students develop the following skills, and get the most out of practice afterwards.

1. Being able to recognize which subject a given problem belongs to.
2. Knowing relevant solving techniques for such type of problems.
3. Being able to choose the most appropriate solution for this particular problem.

There are many tutorial materials available on the website `https://www.mathallstar.org`, including books, videos, articles, and so on.

Meanwhile, it is important to read and understand reference solution instead of just checking the answer. One objective of practice is for students to check whether they understand and master all the necessary solving techniques or not. However, merely obtaining the correct answer does not necessarily mean the most suitable technique is used. Therefore, it is beneficial to understand the solution in addition to obtaining the correct answer.

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Chapter 1

Review

1.1 MOD Basic

1.1.1 Definition

Let a, b, and m be three integers. If the difference of a and b is a multiple of m, we say *a is congruent to b modulo m* and write this relationship as

$$a \equiv b \pmod{m}$$

For example,

$$30 \equiv 16 \equiv 2 \equiv -5 \pmod{7}$$

It is important to note that a and/or b can be negative. In fact, using negative number is a frequently used technique to simplify computation (e.g. see *Section 1.3.1 The Negative One Method* on *page 3*).

1.1.2 Residue System

Given a modulo m, any set of m integers among which no two are congruent to each other is called a residue system modulo m. There are an infinite number of residue systems modulo m. The following one is called the least residue system:

$$\{0, 1, \cdots, m-1\}$$

1.2 MOD Operations

1.2.1 Basic Properties

MOD operations follow the same basic rules as regular algorithmic except the division operation.

Theorem 1.2.1 MOD Addition and Subtraction

Let $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a \pm c \equiv b \pm d \pmod{m}$$

Theorem 1.2.2 MOD Multiplication

Let $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a \times c \equiv b \times d \pmod{m}$$

Theorem 1.2.3 MOD Multiplication with Constant

Let $a \equiv b \pmod{m}$ and k be an integer, then

$$k \times a \equiv k \times b \pmod{m}$$

Theorem 1.2.4 MOD Exponentiation

Let $a \equiv b \pmod{m}$ and k be a positive integer, then

$$a^k \equiv b^k \pmod{m}$$

However, it is important to note the division is an exception here. In order to make the division hold, the relationship between the divisor k and the modulo m must be taken into consideration.

Theorem 1.2.5 MOD Division

Let $a \equiv b \pmod{m}$ and k be an integer, then

$$\frac{a}{k} \equiv \frac{b}{k} \pmod{\frac{m}{\gcd(m,k)}}$$

where $\gcd(m, k)$ is the greatest common divisor of m and k.

For example, given $30 \equiv 6 \pmod 8$, simply dividing both sides by 2 will invalidate the relationship: $15 \not\equiv 3 \pmod 8$. Instead, it is necessary to divide the modulo by 2 which is a common divisor of 8 and 2: $15 \equiv 3 \pmod 4$.

1.2.2 Modular Multiplicative Inverse

Given an integer a, its modular multiplicative inverse modulo m, is also an integer which is written at a^{-1} or $\frac{1}{a}$ and satisfies $aa^{-1} \equiv 1 \pmod m$.

Theorem 1.2.6 Existence of Modular Multiplicative Inverse

The modulo multiplicative inverse of a modulo m exists if and only if a and m are co-prime.

If a^{-1} exists, it must be one of $1, 2, \cdots, m-1$. It can also be shown that the modular multiplicative inverse is unique within this range. Therefore, for a small m, the easiest way to find a^{-1} is just to go through all the values within this range. For example, $\frac{1}{3} \equiv 5 \pmod 7$ because $3 \times 5 \equiv 1 \pmod 7$. No other integers within $[1, 6]$ satisfies this condition.

1.3 Evaluating MOD Expression

Most MOD expressions to be evaluated are in exponential forms. Therefore, most techniques involve exponentiation manipulation.

1.3.1 The Negative One Method

Converting the base to (-1) is a useful technique. Let's review an example.

Example 1.3.1

Find all positive integer n such that $2^n + 1$ is divisible by 3.

Solution

The solution is the set of all odd integers because

$$2^n + 1 \equiv (-1)^n + 1 \equiv 0 \pmod 3$$

Done.

Sometimes, it may require a few intermediate steps to obtain (-1). For example,

$$3^{2018} \equiv (3^2)^{1009} \equiv (-1)^{1009} \equiv -1 \equiv 9 \pmod{10}$$

1.3.2 The Positive One Method

Obtaining $(+1)$ is also a useful technique. In addition to usual elementary transformation to obtain $(+1)$, some challenging problems may require the following two theorems.

Theorem 1.3.1 Fermat Little Theorem

If p is a prime and integer a is not divisible by p, then

$$a^{p-1} \equiv 1 \pmod{p}$$

Theorem 1.3.2 Euler's Theorem

Let a and n be two co-prime integers, then

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

where $\varphi(n)$ is the Euler's *phi* function.

Application of these two theorems will be discussed in another practice by subject series booklet.

1.4 Sum of Digits (The MOD 9 Technique)

When a problem is related to the sum of digits, the following theorem should be considered:

Theorem 1.4.1 Sum of Digits

Let n be a positive integer and $S(n)$ be the sum of its digits, then

$$n \equiv S(n) \pmod{9}$$

The 18^{th} problem in 2017 AMC12A is a typical example.

Example 1.4.1

Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. Which of the following could be the value of $S(n+1)$?

(A) 1 (B) 3 (C) 12 (D) 1329 (E)1265

Solution

By the MOD 9 technique, we have $n \equiv S(n) \equiv 1274 \equiv 5 \pmod 9$. Therefore $n + 1 \equiv 6 \pmod 9$. Among the given choices, only 1329 satisfies this constraint.

Done.

Another type of problems which can be solved using this technique relates to finding a missing digit while all the other digits present. An example is shown below:

Example 1.4.2

The number 2^{29} is a nine-digit number whose digits are all distinct. Which digit of 0 to 9 does not appear?

Solutions to this example is provided in the solution chapter.

1.5 Finding End Digit(s)

1.5.1 Finding the Units Digit

Finding the units digit of m^n is usually easy. The result obviously only depends on the units digit of m. An elementary method to find the last digit is just to observe the repeating pattern. For example, the last digits of 7^n where $n = 1, 2, \cdots$ are

$$7, 9, 3, 1, 7, 9, 3, 1, \cdots$$

Meanwhile, it is always possible to find the answer by evaluating $m^n \pmod{10}$ using the usual techniques.

1.5.2 Finding the Last k Digits

Finding the last k digits of m^n is equivalent to evaluating $m^n \pmod{10^k}$. In addition to the usual evaluation techniques, binomial expansion is a powerful method to find the last k digits when m ends with one[1].

Let's consider an example which is based on a 2011 AMC10 problem.

Example 1.5.1

What are the last 3 digits of 2011^{2011}?

Solution

Firstly, $2011^{2011} \equiv 11^{2011} \pmod{1000}$. Next, let's rewrite 11 as $(10+1)$ and expand the expression:

$$(10+1)^{2011} = \cdots + C_{2011}^3 \times 10^3 + C_{2011}^2 \times 10^2 + C_{2011}^1 \times 10 + 1$$

It is clear that all the terms except that last three will be multiples of 1000 whose last 3 digits are always 000. Therefore, it is sufficient to calulate the last three digits of the sum of the last three terms in order to find the answer.

$$\begin{aligned}
& C_{2011}^2 \times 10^2 + C_{2011}^1 \times 10 + 1 \\
= & \frac{2011 \times 2010}{2} \times 100 + 2011 \times 10 + 1 \\
\equiv & 2011 \times 2010 \times 50 + 11 \times 10 + 1 \\
\equiv & 11 \times 10 \times 50 + 110 + 1 \\
\equiv & 611 \pmod{1000}
\end{aligned}$$

Hence, the last three digits are $\boxed{611}$.

Done.

1.5.3 A Quick Way to Find the Tens Digit of m^n

In addition to the general methods of finding the last k digits of m^n, there exists a quick way to find its last two digits. Because the units digit is alway easy to determine, the key is to find out the tens digit.

m ends with 0 or 5

When m ends with 0, then the tens digit of m^n is always 0. When m ends with 5, it can be shown that m^n can only end with 25 or 75. These two cases are trivial. No special trick is required.

m ends with 1

This is the simplest case which can be solved which a special trick as follows:

[1]For those readers who are not familiar with binomial expansion, this section can be safely ignored

Theorem 1.5.1 Finding tens digit of m^n when m ends with 1

Taking the tens digit of m and multiplying it with the units digit of n, their product's units digit is the tens digit of m^n.

Example 1.5.2

What is the tens digit of 321^{123}?

Solution

The tens digit of 321 is 2. The units digit of 123 is 3. Therefore, the tens digit of 321^{123} is $2 \times 3 = \boxed{6}$.

Done.

In another word, the last two digits of 321^{123} is 61, or $321^{123} \equiv 61 \pmod{100}$.

Theorem 1.5.1 is the steppingstone of finding the tens digit when m ends with other numbers. This theorem can be proved using binomial expansion which is described in the previous section.

m ends with $3, 7$, or 9

These three cases can be transformed to the previous case where m ends with 1 by noting that 3^4, 7^4, and 9^2 all end with 1. Here is an example.

Example 1.5.3

Find the last two digits of 123^{321}.

Solution

Firstly, we note that

$$123^{321} \equiv 23^{321} = (23^4)^{80} \times 23^1 \pmod{100}$$

Because 23^4 ends with 1, the units digit of $(23^4)^{80}$ must be 1. By applying *Theorem 1.5.1*, regardless of the tens digit of 23^4, the tens digit of $(23^4)^{80}$ must be 0 because the units digit of the exponent is 0. In another word, $(23^4)^{80} \equiv 01 \pmod{100}$. Setting this to the above relation leads to

$$123^{321} \equiv 1 \times 23 = \boxed{23} \pmod{100}$$

Done.

It is important to note that 3, 7, and 9 all relate to certain powers of 3:

$$3^1 = 3, 3^2 = 9, 3^3 = 27$$

Therefore, the technique discussed in this section can help to determine the tens digit of a number in the form of 3^k.

A useful fun fact to remember is that $7^4 = 2401$ which ends with 01. Hence,

Let n be a multiple of 4, then the last two digits of 7^n must be 01.

Other Cases

Handling the rest cases requires a bit maneuver. The essential technique is to first factorize m and then process each part separately. A prime factor can only be 2 or ends with 1, 3, 5, 7, or 9. We have already discussed those cases when m ends with an odd digit. Therefore, the only scenario left to tackle is 2^k where k is a positive integer.

The technique to find the tens digit of 2^k depends on the following facts:

- $2^{10} = 1024$, ends with 24
- $24^2 = 576$, ends with 76
- Any power of 76 always ends with 76 itself

This means that if k is sufficiently large, 2^k can be rewritten as a product of a smaller power of 2 and 76^p, modulo 100.

Let's review an example.

Example 1.5.4

Determine the last two digits of 312^{123}.

Solution

First, let's factorize the expression to:

$$312^{123} \equiv 12^{123} = (2^2 \times 3)^{123} = 2^{246} \times 3^{123} \pmod{100}$$

and tackle the two terms separately:

$$2^{246} = (2^{10})^{24} \times 2^6 = (1024)^{24} \times 64 \equiv 24^{24} \times 64 \equiv (24^2)^{12} \times 64 \equiv 76 \times 64 \pmod{100}$$
$$3^{123} = (3^4)^{30} \times 3^3 = 81^{30} \times 27 \equiv 01 \times 27 = 27 \pmod{100}$$

Therefore, the answer we are looking is

$$64 \times 76 \times 27 \equiv \boxed{28} \pmod{100}$$

Done.

1.5.4 Useful Facts

- 25 and 76 are the only two-digit integers whose powers always end with themselves.
- 376 and 625 are the only three-digit integers whose powers always end with themselves.

Chapter 2

Practice

Practice 1

What is the units digit of 13^{2012}?

(ref: 1186 - AMC8)

Practice 2

What is the tens digit of $2015^{2016} - 2017$?

(ref: 2913 - AMC10)

Practice 3

Let $a > b > c$ be three positive integers. If their remainders are 2, 7, and 9 respectively when being divided by 11. Find the remainder when $(a+b+c)(a-b)(b-c)$ is divided by 11.

(ref: 122)

Practice 4

What is the last digit of 9^{2015}?

(ref: 272)

Practice 5

What are the last two digits of 8^{88}?

(ref: 273)

Practice 6

Find the remainder when $3^{2015} + 4^{2015}$ is divided by 5?

(ref: 274)

Practice 7

Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99 \cdots 9}_{999}$ is divided by 1000.

(ref: 280 - AIME)

Practice 8

The number 2^{29} is a nine-digit number whose digits are all distinct. Which digit of 0 to 9 does not appear?

(ref: 311)

Practice 9

Let four positive integers a, b, c, and d satisfy $a + b + c + d = 2015$. Prove $a^3 + b^3 + c^3 + d^3$ cannot be an even number.

(ref: 1120)

Practice 10

What is the tens digit of 7^{2011}?

(ref: 1171 - AMC8)

Practice 11

What are the sign and units digit of the product of all the odd negative integers strictly greater than -2015?

(ref: 1259 - AMC10)

Practice 12

What is the tens digit in the sum $7! + 8! + 9! + ... + 2006!$

(ref: 1721 - AMC10)

Practice 13

What is the units digit of the product $7^{23} \times 8^{105} \times 3^{18}$?

(ref: 1819 - MathCounts)

Practice 14

Determine the units digit of the sum $0! + 1! + 2! + \cdots + n! + \cdots + 20!$?

(ref: 2654 - BCML)

Practice 15

What is the last digit of 7^{222}?

(ref: 2741)

Practice 16

Let n be any positive integer, show that

$$(5n+1)(5n+2)(5n+3)(5n+4) \equiv -1 \pmod{25}$$

(ref: 4160)

Practice 17

Let m be the least positive integer divisible by 17 whose digits sum is 17. Find m.

(ref: 70 - AIME)

Practice 18

The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc.

(ref: 90 - AIME)

Practice 19

Let integer a, b, and c satisfy $a+b+c=0$, prove $|a^{1999}+b^{1999}+c^{1999}|$ is a composite number.

(ref: 310)

Practice 20

What are the last two digits in the sum of the factorials of the first 100 positive integers?

(ref: 1117)

Practice 21

Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

(ref: 1608 - AMC10)

Practice 22

Let $f(n)$ denote the sum of the digits of n. Find $f(f(f(4444^{4444})))$.

(ref: 2216 - IMO)

Practice 23

What is the last digit of $17^{17^{17^{17}}}$?

(ref: 2540 - PUMaC)

Practice 24

Does there exist a polynomial $P(x)$ such that $P(1) = 2015$ and $P(2015) = 2016$?

(ref: 3971)

Practice 25

Find all prime number p such that both $4p^2 + 1$ and $6p^2 + 1$ are prime numbers.

(ref: 174 - Poland)

Practice 26

The number 2017 is prime. Let $S = \sum_{k=0}^{62} \binom{2014}{k}$. What is the remainder when S is divided by 2017?

(ref: 474 - AMC12)

Practice 27

The number obtained from the last two non-zero digits of 90! is equal to n. What is n?

(ref: 1508 - AMC10)

Practice 28

If for any integer $k \neq 27$ and $\left(a - k^{2015}\right)$ is divisible by $(27 - k)$, what is the last two digits of a?

(ref: 2621)

Chapter 3

Solution

Practice 1

What is the units digit of 13^{2012}?

(ref: 1186 - AMC8)

Method 1

We note that the ending digit of 3^k repeats every 4 terms: $3, 9, 7, 1, 3, \cdots$. Because 2012 is a multiple of 4, therefore the ending digits of 13^{2012} is the same as 3^4 which is $\boxed{1}$.

Method 2 – *Power of* (-1)

$$13^{2012} \equiv 3^{2012} \equiv 9^{1006} \equiv (-1)^{1006} \equiv \boxed{1} \pmod{10}$$

Practice 2

What is the tens digit of $2015^{2016} - 2017$?

(ref: 2913 - AMC10)

Firstly, $2015^{2016} \equiv 15^{2016} \pmod{100}$.

It is easy to verify that $(k > 1)$, $15^k \equiv 25 \pmod{100}$ when k is even, and $15^k \equiv 75 \pmod{100}$ when k is odd.

Therefore 15^{2016} will end with 25 which leads to the final answer as $\boxed{8}$.

Practice 3

Let $a > b > c$ be three positive integers. If their remainders are 2, 7, and 9 respectively when being divided by 11. Find the remainder when $(a+b+c)(a-b)(b-c)$ is divided by 11.

(ref: 122)

$$(a+b+c)(a-b)(b-c) \equiv (2+7+9)(2-7)(7-9) = 180 \equiv \boxed{4} \pmod{11}$$

Practice 4

What is the last digit of 9^{2015}?

(ref: 272)

$$9^{2015} \equiv (-1)^{2015} \equiv -1 \equiv \boxed{9} \pmod{10}$$

Practice 5

What are the last two digits of 8^{88}?

(ref: 273)

$$8^{88} \equiv 2^{264} \equiv \left(2^{10}\right)^{26} \times 2^4 \equiv 24^{26} \times 16 \equiv 76^{13} \times 16 \equiv 76 \times 16 \equiv \boxed{16} \pmod{100}$$

Practice 6

Find the remainder when $3^{2015} + 4^{2015}$ is divided by 5?

(ref: 274)

$$3^{2015} + 4^{2015} \equiv (3^2)^{1007} \times 3 + (-1)^{2015} \equiv (-1)^{1007} \times 3 - 1 \equiv \boxed{1} \pmod{5}$$

Practice 7

Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{999}$ is divided by 1000.

(ref: 280 - AIME)

Note that $\underbrace{99\cdots9}_{k} \equiv 999 \equiv -1 \pmod{1000}$ when $k \geq 3$. The original expression is congruent to

$$9 \times 99 \times \underbrace{(-1)(-1)\cdots(-1)}_{997} \equiv 891 \times (-1) \equiv \boxed{109} \pmod{1000}$$

Practice 8

The number 2^{29} is a nine-digit number whose digits are all distinct. Which digit of 0 to 9 does not appear?

(ref: 311)

By the divide by 9 technique, the sum of these digits must be congruent to $2^{29} \pmod 9$ which is (-4) (see below). Hence, the missing digit is $\boxed{4}$.

$$2^{29} \equiv (2^3)^9 \times 2^2 \equiv (-1)^9 \times 4 \equiv -4 \pmod 9$$

Practice 9

Let four positive integers a, b, c, and d satisfy $a + b + c + d = 2015$. Prove $a^3 + b^3 + c^3 + d^3$ cannot be an even number.

(ref: 1120)

It is easy to show that regardless of integer n's parity, it always hold that $n^3 \equiv n \pmod 2$ because any power of n will change change odd even parity. Therefore,

$$a^3 + b^3 + c^3 + d^3 \equiv a + b + c + d \equiv 2015 \equiv 1 \pmod 2$$

Practice 10

What is the tens digit of 7^{2011}?

(ref: 1171 - AMC8)

$$7^{2011} = (7^4)^{502} \times 7^3 \equiv 01^{502} \times 43 \equiv 43 \pmod{100}$$

Hence the answer is $\boxed{D} = 4$.

Practice 11

What are the sign and units digit of the product of all the odd negative integers strictly greater than -2015?

(ref: 1259 - AMC10)

There are odd number of negative integers, therefore the sign is negative. Clearly, there is (-5) among them and all the other numbers are odd, hence, the units digit must be 5.

Practice 12

What is the tens digit in the sum $7! + 8! + 9! + ... + 2006!$

(ref: 1721 - AMC10)

When $k \geq 10$, $k!$ must be multiple of 100 whose last two digits will be 0. Therefore, the desired results is the same as the tens digit of $7!+8!+9!$. We can compute $7! = 5040$, therefore

$$7! + 8! + 9! \equiv 40 + 40 \times 8 + 40 \times 8 \times 9 \equiv 40 + 20 + 80 \equiv 40 \pmod{100}$$

Practice 13

What is the units digit of the product $7^{23} \times 8^{105} \times 3^{18}$?

(ref: 1819 - MathCounts)

The last digit of 7^k repeats as $7, 9, 3, 1, 7, \cdots$. Therefore 7^{23} ends with 3.

The last digit of 8^k repeats as $8, 4, 2, 6, 8, \cdots$. Therefore 8^{105} ends with 8.

The last digit of 3^k repeats as $3, 9, 7, 1, 3, \cdots$. Therefore 3^{18} ends with 9.

Hence, the units digit of $7^{23} \times 8^{105} \times 3^{18}$ is the same as the last digit of $3 \times 8 \times 9$ which is 6.

Practice 14

Determine the units digit of the sum $0! + 1! + 2! + \cdots + n! + \cdots + 20!$?

(ref: 2654 - BCML)

Where $n \geq 5$, $n!$ must be a multiple of 10 whose units' digit will be 0. Therefore, the original question is equivalent to finding the unit digit of

$$0! + 1! + 2! + 3! + 4! = 1 + 1 + 2 + 6 + 24 = 34$$

Therefore, the answer is $\boxed{4}$.

Practice 15

What is the last digit of 7^{222}?

(ref: 2741)

The last digit of 7^n rotates in the set of $\{7, 9, 3, 1\}$. Because $2222 \equiv 2 \pmod 4$, we find the answer is $\boxed{9}$.

Practice 16

Let n be any positive integer, show that

$$(5n+1)(5n+2)(5n+3)(5n+4) \equiv -1 \pmod{25}$$

(ref: 4160)

$$\begin{aligned}(5n+1)(5n+2)(5n+3)(5n+4) &= ((5n+1)(5n+4))((5n+2)(5n+3)) \\ &= (25n^2+5n+4)(25n^2+5n+6) \\ &= (25n^2+5n)^2 + 10 \times (25n^2+5n) + 24 \\ &\equiv 24 \\ &\equiv -1 \pmod{25}\end{aligned}$$

Practice 17

Let m be the least positive integer divisible by 17 whose digits sum is 17. Find m.

(ref: 70 - AIME)

Let $s(m)$ be the digit sum of m. It must hold that

$$m \equiv s(m) \equiv 17 \equiv -1 \pmod 9$$

Because m is a multiple of 17, let $m = 17k$ which means

$$17k \equiv -1 \pmod 9$$

Obviously $k = 1$ is one solution, therefore its general solution is $k = 9n + 1$. Try $n = 0, 1, 2, 3, \cdots$ finding $m = \boxed{476}$ is the first positve solution that has $s(m) = 17$.

Practice 18

The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc.

(ref: 90 - AIME)

We know there are only two three-digit number whose squares end with themselves: 625 and 376. Therefore, N must end with either 625 or 376. There are some MOD tricks we can play to narrow down the choices, but it is not too hard to try all the possible thousands digit and find $N = 9376$. Therefore the answer is $\boxed{937}$.

Practice 19

Let integer a, b, and c satisfy $a+b+c = 0$, prove $|a^{1999}+b^{1999}+c^{1999}|$ is a composite number.

(ref: 310)

Let $d = a^{1999} + b^{1999} + c^{1999}$, we are going to show that d is a multiple of 6 which means it is composite.

Firstly, d is a multiple of 2 because

$$d \equiv a^{1999} + b^{1999} + c^{1999} \equiv a + b + c \equiv 0 \pmod 2$$

Next, it is easy to show that $x^3 \equiv x \pmod 3$. This is because $x^3 - x = (x-1)x(x+1)$ which is a product of three consecutive integers. One of them must be a multiple of 3. Hence $x^3 - x \equiv 0 \pmod 3$. It follows that

$$\begin{aligned} d &\equiv a \cdot a^{1998} + b \cdot b^{1998} + c \cdot c^{1998} \\ &\equiv a \cdot a^{666} + b \cdot b^{666} + c \cdot c^{666} \\ &\equiv a \cdot a^{222} + b \cdot b^{222} + c \cdot c^{222} \\ &\equiv a \cdot (a^{74})^3 + b \cdot (b^{74})^3 + c \cdot (c^{74})^3 \\ &\equiv a^{75} + b^{75} + c^{75} \\ &\equiv a + b + c \\ &\equiv 0 \pmod 3 \end{aligned}$$

This means d is a multiple of 3.

Practice 20

What are the last two digits in the sum of the factorials of the first 100 positive integers?

(ref: 1117)

When $k \geq 10$, $k!$ must be a multiple of 100 because its prime factorization contains two 5s and more than two 2s. Therefore, the desired result equals the last two digits of

$$1! + 2! + \cdots + 9! \pmod{100}$$

Let's compute these terms separately:

$$\begin{aligned} 1! &\equiv 1 \pmod{100} \\ 2! &\equiv 2 \pmod{100} \\ 3! &\equiv 6 \pmod{100} \\ 4! &\equiv 24 \pmod{100} \\ 5! &\equiv 20 \pmod{100} \\ 6! &\equiv 20 \pmod{100} \\ 7! &\equiv 40 \pmod{100} \\ 8! &\equiv 20 \pmod{100} \\ 9! &\equiv 80 \pmod{100} \end{aligned}$$

Adding the numbers on the right side leads to the result $\boxed{13}$.

Practice 21

Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

(ref: 1608 - AMC10)

Firstly, 2008^2 must end with 4. Meanwhile the units digit of 2^k repeats every 4 numbers: $2, 4, 8, 6, 2, \cdots$. This means that the end digit of 2^{2008} must end with 6. Therefore, k ends with 0 which implies k^2 ends with 0.

To determine the last digit of 2^k, it is sufficient to compute $k \pmod 4$ because we can utilize the repeating pattern observed above. It is easy to see that $k \equiv 0 \pmod 4$. Therefore, 2^k will end with 6.

Hence $k^2 + 2^k$ ends with $\boxed{6}$.

Practice 22

Let $f(n)$ denote the sum of the digits of n. Find $f(f(f(4444^{4444})))$.

(ref: 2216 - IMO)

This is a typical problem that can be solved by the MOD-by-9 method.

Because $4444^{4444} < 10000^{4444} = 10^{17776}$, we find that 4444^{4444} has at most 17776 digits, which means that $f(4444^{4444})$ can not be greater than $9 \times 17776 = 159984$. It follows that $f(f(4444^{4444}))$ can not be greater than $9 \times 5 = 45$. Similarly, $f(f(f(4444^{4444})))$ can not be greater than $3 + 9 = 12$.

Meanwhile, we have

$$f(f(f(4444^{4444}))) \equiv 4444^{4444} \pmod 9$$

and

$$4444^{4444} \equiv (-2)^{4444} = 2^{4444} = 2^{4440} \times 2^4 = 64^{740} \times 16 \equiv 1^7 \times 7 \equiv 7 \pmod 9$$

There is only one positive integer no greater than 12 which is congruent to 7 modulo 9. Hence, the answer is $\boxed{7}$.

Practice 23

What is the last digit of $17^{17^{17^{17}}}$?

(ref: 2540 - PUMaC)

We know that the last digit of 17^k repeats as 7, 9, 3, 1, 7, $\cdots$. Therefore it is sufficient to compute $17^{17^{17}} \pmod 4$ which is

$$17^{17^{17}} \equiv 1^{17^{17}} \equiv 1 \pmod 4$$

Therefore, the final answer is $\boxed{7}$.

Practice 24

Does there exist a polynomial $P(x)$ such that $P(1) = 2015$ and $P(2015) = 2016$?

(ref: 3971)

Firstly, we know that if k is odd, then for any integer m, we must have $km \equiv m \pmod 2$.

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. Then

$$P(2015) \equiv a_n + a_{n-1} + \cdots + a_1 + a_0 = P(1) = 2015 \equiv 1 \pmod 2$$

But 2016 is even, Therefore it is impossible for $P(2015) = 2016$.

Practice 25

Find all prime number p such that both $4p^2 + 1$ and $6p^2 + 1$ are prime numbers.

(ref: 174 - Poland)

When $p = 5$, both $4p^2 + 1 = 101$ and $6p^2 + 1 = 151$ are prime. Therefore $p = 5$ is one solution. We are going to show that this is the only solution.

When $p \equiv \pm 1 \pmod 5$, we have $4p^2 + 1 \equiv 0 \pmod 5$. This means that $(4p^2 + 1)$ is a multiple of 5 which cannot be prime.

Meanwhile, when $p \equiv \pm 2 \pmod 5$, we have $6p^2 + 1 \equiv 0 \pmod 5$. This means that $(6p^2 + 1)$ is a multiple of 5 which is not prime.

Practice 26

The number 2017 is prime. Let $S = \sum_{k=0}^{62} \binom{2014}{k}$. What is the remainder when S is divided by 2017?

(ref: 474 - AMC12)

First, let's simplify C_{2014}^k as

$$\begin{aligned}
C_{2014}^k &\equiv \frac{2014 \times 2013 \times \cdots (2014 - k + 1)}{k!} \\
&\equiv \frac{(-3)(-4) \cdots (-k - 2}{k!)} \\
&\equiv (-1)^k C_{k+2}^k \\
&\equiv (-1)^k C_{k+2}^2
\end{aligned}$$

$$\equiv (-1)^k \times \frac{(k+2)(k+1)}{2}$$

When k is even, let $k = 2m$. Then the sum of those "even" terms in the above expression equals

$$\sum_{m=0}^{31} \frac{(2m+2)(2m+1)}{2} = \sum_{m=0}^{30} (2m^2 + 3m + 1)$$

When k is odd, let $k = 2m+1$. Then the sum of those "odd" terms above equals

$$\sum_{m=0}^{30} (-1) \times \frac{(2m+3)(2m+2}{2} = -\sum_{m=0}^{30} (2m^2 + 5n + 3)$$

Adding them together we have

$$S = 2 \times 31^2 - 2\sum_{m=0}^{30} m + 3 \times 31 + 32 - 3 \times 31 = \boxed{1024}$$

Practice 27

The number obtained from the last two non-zero digits of 90! is equal to n. What is n?

(ref: 1508 - AMC10)

First, the number of trailing zero equals the number of divisor 5 that 90! has. This equals

$$\left\lfloor \frac{90}{5} \right\rfloor + \left\lfloor \frac{90}{5^2} \right\rfloor = 21$$

This means that 90! has 21 trailing zeros. Let $N = \frac{90!}{10^{21}}$. Then the desired answer n equals $N \pmod{100}$.

Clearly, N still has more than two divisors of 2. Hence, $N \equiv 0 \pmod 4$. In order to calculate $N \pmod{100}$, we just need to compute $N \pmod{25}$.

By the conclusion of practice 16, we know $(5k+1)(5k+2)(5k+3)(5k+4) \equiv -1 \pmod{25}$. In order to use this conclusion, let

$$M = 1 \times 2 \times 3 \times 4 \times \underline{1} \times 6 \times \cdots \times 86 \times 87 \times 88 \times 89 \times \underline{8}$$

That is, to eliminate all 5 from 90!, i.e. $5 \to 1$, $10 \to 2$, $\cdots$, $25 \to 1$, $\cdots$, $90 \to 18$. Then, we have

$$\begin{aligned} M = & \ 1 \times 2 \times 3 \times 4 \times 1 \times 6 \times \cdots \times 86 \times 87 \times 88 \times 89 \times 18 \\ = & \ (1 \times 2 \times 3 \times 4)(6 \times 7 \times 8) \cdots (86 \times 87 \times 88 \times 89) \\ & \ (1 \times 2 \times 3 \times 4)(6 \times 7 \times 8 \times 9) \cdots (16 \times 17 \times 18) \\ & \ (1 \times 2 \times 3) \end{aligned}$$

The 2^{nd} last line above is corresponding to all the numbers which divides 5, but not 25. The last line is corresponding to those numbers which are multiple of 25. Therefore

$$M \equiv (-1)^{10} \times (-1)^3 \times (16 \times 17 \times 19)(1 \times 2 \times 3) \equiv 24 \pmod{25}$$

Meanwhile, we have $2^{21} \equiv (2^{10})^2 \times 2 \equiv (-1)^2 \times \ 2 \equiv 2 \pmod{25}$. It follows that

$$N = \frac{M}{2^{21}} \equiv \frac{24}{2} \equiv 12 \pmod{25}$$

Together with the fact of $N \equiv 0 \pmod 4$, we found $N \equiv \boxed{12} \pmod{100}$

Practice 28

If for any integer $k \neq 27$ and $\left(a - k^{2015}\right)$ is divisible by $(27 - k)$, what is the last two digits of a?

(ref: 2621)

Let $f(k) = a - k^{2015}$. Because $(k - 27) \mid f(k)$, we have $f(27) = 0$, i.e. 27 is a root of $f(k)$. This means that $a - 27^{2015} = 0$ which implies that the last two digits of a is the same as those of 27^{2015}.

$$27^{2015} = 3^{6045} = (3^4)^{1511} \times 3 = 81^{1511} \times 3$$

Now by applying the "quick way to find the tens digit" trick, we know the tens digit of 81^{1511} is 8. Meanwhile, its units digit obviously is 1. Hence, 81^{1511} ends with 81 which leads to 27^{2015} ends with $\boxed{43}$.