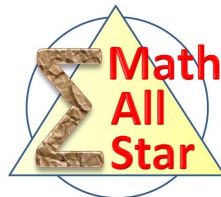


---

# Geometry

## Coordinate Basics

---



*Math for Gifted Students*

<http://www.mathallstar.org>




# Coordinate Basics



## Instructions

- Write down and submit intermediate steps along with your final answer.
- If the final result is too complex to compute, give the expression. e.g.  $C_{100}^{50}$  is acceptable.
- Problems are not necessarily ordered based on their difficulty levels.
- Always ask yourself what makes this problem a good practice?
- Read through the reference solution even if you can solve the problem for additional information which may help you to solve this type of problems.

## Legends

-  *Tips, additional information etc*
-  *Important theorem, conclusion to remember.*
-  *Addition questions for further study.*

## My Comments and Notes

# Coordinate Basics



The emphasis of this practice is to write the required equations or formulas **directly**.



You should remember all the conclusions in this practice.

## Coordinate Basics



## Practice 1

**(Distance)** What is the distance between  $(x_1, y_1)$  and  $(x_2, y_2)$ ?

## Practice 2

**(Middle Point)** What is the coordinate of the middle point between  $(x_1, y_1)$  and  $(x_2, y_2)$ ?

## Practice 3

**(Centroid)** Give a triangle whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ , respectively. What is the coordinate of its center of mass (centroid)?

## Practice 4

**(Incenter)** Give a triangle whose vertices are  $A(x_a, y_a)$ ,  $B(x_b, y_b)$ , and  $C(x_c, y_c)$ , respectively. If  $a = BC$ ,  $b = CA$ , and  $c = BA$ , what is the coordinate of its incenter?

## Practice 5

**(Interpolation)** Give two points  $A(x_a, y_a)$  and  $B(x_b, y_b)$ , a point  $C$  on  $\overline{AB}$ . If  $AC : CB = m : n$  where  $m$  and  $n$  are two integers, find the coordinate of point  $C$ .

## Practice 6

**(Line by Two Points)** If a straight line  $l$  passes two distinct points  $(x_1, y_1)$  and  $(x_2, y_2)$ , what is  $l$ 's equation?

## Practice 7

**(Line by Intercepts)** What is the equation of a straight line if its  $x$ -intercept and  $y$ -intercept are  $a$  and  $b$ , respectively?

## Coordinate Basics

**Practice 8**

**(Plane by Intercepts)** What is the equation of a plane if its  $x$ -intercept,  $y$ -intercept, and  $z$ -intercept are  $a$ ,  $b$ , and  $c$ , respectively?

**Practice 9**

**(Point to Line)** What is the distance from the point  $(x_0, y_0)$  to the line  $Ax + By + C = 0$ ?

**Practice 10**

**(Point to Plane)** What is the distance from the point  $(x_0, y_0, z_0)$  to the plane  $Ax + By + Cz + D = 0$ ?

**Practice 11**

**(Triangle Area)** What is the area of a triangle whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ ?

**Practice 12**

**(Polygon Area)** What is the area of a polygon whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $\dots$  and  $(x_n, y_n)$ ?

**Practice 13**

Given two parallel lines:  $Ax + By + C_1 = 0$  and  $Ax + By + C_2 = 0$ , find the locus of all the points that are equidistant to these two lines.

## Coordinate Basics



## Practice 14

**(Circle Tangent Line)** Let point  $P(x_0, y_0)$  be on circle  $O$ , find the equation of the straight line which is tangent to  $O$  on point  $P$  when  $O$  is given by:

(i)  $x^2 + y^2 = r^2$

(ii)  $(x - a)^2 + (y - b)^2 = r^2$

(iii)  $x^2 + y^2 + Dx + Ey + F = 0$

## Practice 15

**(Chord Passing Tangent Points)** Let point  $P(x_0, y_0)$  be outside the circle  $O : x^2 + y^2 = 0$ . If  $PA$  and  $PB$  are two lines that pass  $P$  and are tangent to  $O$  at  $A$  and  $B$ , find the equation of line  $AB$ .

## Practice 16

**(Distance to Tangent Points)** Let point  $P(x_0, y_0)$  be outside the circle  $O$ . If  $PA$  is a line that passes  $P$  and is tangent to  $O$  at  $A$ , find the distance between  $P$  and  $A$  when  $O$  is given by:

(i)  $x^2 + y^2 = r^2$

(ii)  $(x - a)^2 + (y - b)^2 = r^2$

(iii)  $x^2 + y^2 + Dx + Ey + F = 0$

---

# Reference Solutions

---

## Coordinate Basics



## Practice 1

**(Distance)** What is the distance between  $(x_1, y_1)$  and  $(x_2, y_2)$ ?

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

## Practice 2

**(Middle Point)** What is the coordinate of the middle point between  $(x_1, y_1)$  and  $(x_2, y_2)$ ?

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

## Practice 3

**(Centroid)** Give a triangle whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ , respectively. What is the coordinate of its center of mass (centroid)?

$$\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

## Practice 4

**(Incenter)** Give a triangle whose vertices are  $A(x_a, y_a)$ ,  $B(x_b, y_b)$ , and  $C(x_c, y_c)$ , respectively. If  $a = BC$ ,  $b = CA$ , and  $c = BA$ , what is the coordinate of its incenter?

$$\left( \frac{ax_a + bx_b + cx_c}{a + b + c}, \frac{ay_a + by_b + cy_c}{a + b + c} \right)$$



## Coordinate Basics



## Practice 5

**(Interpolation)** Give two points  $A(x_a, y_a)$  and  $B(x_b, y_b)$ , a point  $C$  on  $\overline{AB}$ . If  $AC : CB = m : n$  where  $m$  and  $n$  are two integers, find the coordinate of point  $C$ .

$$\left( \frac{nx_a + mx_b}{m + n}, \frac{ny_a + my_b}{m + n} \right)$$

## Practice 6

**(Line by Two Points)** If a straight line  $l$  passes two distinct points  $(x_1, y_1)$  and  $(x_2, y_2)$ , what is  $l$ 's equation?

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

## Practice 7

**(Line by Intercepts)** What is the equation of a straight line if its  $x$ -intercept and  $y$ -intercept are  $a$  and  $b$ , respectively?

$$\frac{x}{a} + \frac{y}{b} = 1$$

## Practice 8

**(Plane by Intercepts)** What is the equation of a plane if its  $x$ -intercept,  $y$ -intercept, and  $z$ -intercept are  $a$ ,  $b$ , and  $c$ , respectively?

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

## Coordinate Basics



## Practice 9

**(Point to Line)** What is the distance from the point  $(x_0, y_0)$  to the line  $Ax + By + C = 0$ ?

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

## Practice 10

**(Point to Plane)** What is the distance from the point  $(x_0, y_0, z_0)$  to the plane  $Ax + By + Cz + D = 0$ ?

$$\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

## Practice 11

**(Triangle Area)** What is the area of a triangle whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ ?

$$\frac{|(x_1y_2 - y_1x_2) + (x_2y_3 - y_2x_3) + (x_3y_1 - y_3x_1)|}{2}$$

## Practice 12

**(Polygon Area)** What is the area of a polygon whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $\dots$  and  $(x_n, y_n)$ ?

$$\frac{|(x_1y_2 - y_1x_2) + (x_2y_3 - y_2x_3) + \dots + (x_ny_1 - y_nx_1)|}{2}$$

## Coordinate Basics



## Practice 13

Given two parallel lines:  $Ax + By + C_1 = 0$  and  $Ax + By + C_2 = 0$ , find the locus of all the points that are equidistant to these two lines.

$$Ax + By + \frac{C_1 + C_2}{2} = 0$$

## Practice 14

**(Circle Tangent Line)** Let point  $P(x_0, y_0)$  be on circle  $O$ , find the equation of the straight line which is tangent to  $O$  on point  $P$  when  $O$  is given by:

(i)  $x^2 + y^2 = r^2$

(ii)  $(x - a)^2 + (y - b)^2 = r^2$

(iii)  $x^2 + y^2 + Dx + Ey + F = 0$

(i)  $x_0x + y_0y = r^2$

(ii)  $(x_0 - a)(x - a) + (y_0 - b)(y - b) = r^2$

(iii)  $x_0x + y_0y + D \cdot \frac{x_0 + x}{2} + E \cdot \frac{y_0 + y}{2} + F = 0$

## Practice 15

**(Chord Passing Tangent Points)** Let point  $P(x_0, y_0)$  be outside the circle  $O : x^2 + y^2 = r^2$ . If  $PA$  and  $PB$  are two lines that pass  $P$  and are tangent to  $O$  at  $A$  and  $B$ , find the equation of line  $AB$ .

$$x_0x + y_0y = r^2$$

## Coordinate Basics



## Practice 16

**(Distance to Tangent Points)** Let point  $P(x_0, y_0)$  be outside the circle  $O$ . If  $PA$  is a line that passes  $P$  and is tangent to  $O$  at  $A$ , find the distance between  $P$  and  $A$  when  $O$  is given by:

(i)  $x^2 + y^2 = r^2$

(ii)  $(x - a)^2 + (y - b)^2 = r^2$

(iii)  $x^2 + y^2 + Dx + Ey + F = 0$

(i)  $\sqrt{x_0^2 + y_0^2 - r^2}$

(ii)  $\sqrt{(x_0 - a)^2 + (y_0 - b)^2 - r^2}$

(iii)  $\sqrt{x_0^2 + y_0^2 + Dx_0 + Ey_0 + F}$



*Tip: How can you remember these three formulas?*



---

## Battle Field

Selective problems from recent competitions:

- Problem 1: 2016 AMC10B #20 (Ref 2925)
- Problem 2: 2015 AIME II #9 (Ref 76)
- Problem 3: 2015 AIME I #4 (Ref 56)
- Problem 4: 2013 AMC12A #13 (Ref 489)
- Problem 5: 2013 MathCounts State Target #6 (Ref 1841)
- Problem 6: 2012 AMC10B #23 (Ref 1427)
- Problem 7: 2012 MathCounts Chapter Target #3 (Ref 1975)
- Problem 8: 2012 MathCounts Chapter Team #2 (Ref 1982)